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THE ROLE OF SIMULTANEITY IN THE METHODOLOGY OF THE GLOBAL POSITIONING NAVIGATION SYSTEM

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Abstract: The Global Positioning System (GPS) adjusts the rates of proper clocks carried onboard its satellites in order to obtain its high level of accuracy. This procedure was first suggested on the basis of experimental data obtained by Hafele and Keating (ref.) which demonstrate the effects of time dilation on atomic clocks carried onboard circumnavigating airplanes. There has been a longstanding belief among physicists that these clock-rate adjustments are consistent with the predictions of the Lorentz transformation (LT), but the present work shows that this is actually not the case. The experience with GPS makes clear, for example, that there is a definite asymmetry in the clock rates that runs contrary to the LT predictions. The strict proportionality assumed between the rates of proper clocks is also clear evidence of the absolute remote simultaneity of events, which again stands in contradiction to what must be inferred from the LT. An alternative version of the Lorentz transformation exists which takes into account the above proportionality relation (t'=t/Q) while still satisfying both of Einstein's postulates of relativity. It can rightly be called the GPS-LT because it is completely consistent with the adjusted clock-rate procedure. On this basis a method is devised for modifying the rates of proper atomic clocks onboard orbiting satellites as a function of their position and velocity relative to the earth's center of mass (ECM) so as to insure that they continuously run at the same rate as a standard clock on the earth's surface.

Keywords: Lorentz transformation (LT); asymmetric time dilation; relativistic velocity transformation (RVT); Global Positioning System (GPS); alternative Lorentz transformation (ALT); uniform scaling of coordinates

1. Introduction:

In the introductory remarks of their landmark paper on the rates of atomic clocks carried onboard circumnavigating airplanes, Hafele and Keating [1-2] pointed out that there was great uncertainty at that time about "predicted time differences between traveling and reference clocks after a round trip." This summary makes clear that it is not possible to come to a definitive conclusion about the so-called "clock paradox" purely on the basis of considerations of the Lorentz transformation (LT) of Einstein's Theory of Special Relativity (STR) [3]. Hafele and Keating determined that their observed results could be reconciled with the LT by assuming that the slowing down of the atomic clocks in flight was strictly a function of their speed relative to a specific reference of "hypothetical coordinate clocks of an underlying nonrotating (inertial) space." They included a footnote in which they argued that the time difference of two clocks located at the same place is a "physically observable quantity that is invariant," making in their view a "subsequent coordinate transformation into the noninertial rest system for the clocks … unnecessary." This rather complicated justification still leaves open some questions about whether the LT is actually consistent with their observed timing results, as will be discussed in the following section.

2. Contradictions in the LT Predictions for Time Dilation:

The main application of relativity theory in the GPS methodology is the adjustment of clocks onboard satellites so that they run at the same rate as clocks at some point on the earth's surface. Superficially, this adjustment can be said to be consistent with the LT prediction of time dilation. There are at least two aspects of this procedure that bear closer attention, however. For one thing, the LT clearly predicts that there is a symmetric relationship between the rates of clocks in relative motion. Accordingly, a clock in motion relative to an observer always runs slower than his local proper clock. It is therefore a matter of perspective which of two clocks is actually running slower. The same symmetric relationship is believed to exist for other properties such as the lengths of objects and their inertial masses. The other key aspect of STR is that remote non-simultaneity of events is possible. The latter conclusion follows directly from the time equation of the LT given below [$\gamma=(1-v^2c^{-2})^{-0.5}$]:

$$t' = \gamma\,(t - vxc^{-2}). \tag{1}$$

In this equation t and t' are the respective measured values for the time on an event by observers in different inertial rest frames S and S' and x is the corresponding location of the event as measured by a stationary observer in S (v is the relative speed of S and S' directed along the common x-x' axis and c=2.99792458 ms^{-1} is the speed of light in free space). The two coordinate systems are chosen to coincide at their respective origins (x=x'=0) at t=t'=0. A related equation expressed in terms of intervals $\Delta x=x_2-x_1$ and $\Delta t=t_2-t_1$ for two events will also prove useful in the present discussion:

$$\Delta t' = \gamma\,(\Delta t - v\Delta xc^{-2})= \gamma\,\eta^{-1}\,\Delta t, \tag{2}$$

with $\eta = (1 - vc^{-2}\Delta x/\Delta t)^{-1}$. The latter relation has the advantage of being independent of the choice of coordinate system. It is clear from both eqs. (1-2) that t (Δt) can be equal to zero without the same being true for t' (Δt') so long as both v and x (Δx) do not vanish as well. This is the justification for making the above claim of remote non-simultaneity of events based on the LT. This theoretical possibility was first proposed by Poincaré in 1898 [4, 5]. He made the point that the assumption of remote simultaneity had never been confirmed in actual experimental investigations. One should add that a clear manifestation of remote non-simultaneity also did not exist at that time.

The motivation for adjusting the rates of satellite clocks is quite simple [6, 7]. One would like to measure the distance L between the satellite and a position on the earth's surface. This can be done in practice by measuring the elapsed time Δt for a light pulse to traverse that distance (a correction for the relative speed of the satellite and the earth at the time of emission also must be made). This result can then be converted into the desired value for L by using the definition of distance as the product of the speed and the elapsed time necessary for a light pulse to travel between the corresponding two endpoints, i.e. L=c Δt in the present case. For a suitably accurate determination to be made for Δt, it is obvious that the two local clocks used to measure both the time of emission of the light pulse and its arrival at the other point must run at exactly the same rate. Because of time dilation, however, it is clear that the latter requirement is not satisfied when only proper clocks are used for this measurement. A correction for the effects of gravity on clock rates is also required [7].

The calculation to determine the amount of the clock-rate adjustment is quite simple. The Hafele-Keating study [1-2] showed that the rates of clocks slow down in direct proportion to γ(v), where v is the speed of the clock relative to the earth's center of mass (ECM). Since the earth is rotating beneath each clock, one flying eastward is slowed down more because of time dilation than its counterpart at the airport of departure, which in turn is slower than a clock moving in the westerly direction. A computational procedure for determining the amount of these adjustments will be considered in the next section, but there is a more fundamental point to be discussed with regard to how this procedure is related to the predictions of the LT. It clearly shows that, when the effects of gravity are disregarded, *the clock on the satellite runs objectively slower than the one on the earth's surface.* This means that the symmetry expected from the LT does not occur in practice. Both the observer on the satellite and his counterpart on the ground agree on the simple fact of which clock runs slower.

The key question to be answered then is how this indisputable experimental result can be reconciled with the predictions of the LT. One might argue that the LT is not directly applicable in this situation because the satellite clocks are being acted upon continuously by external forces. Experiments with muon decay [8] speak strongly against this conclusion, however, because they have shown that the amount of time dilation is determined solely by the speed of the particles relative to the laboratory and are in no way affected by the degree of acceleration they experience. Simply balancing all external forces on both the satellite clocks and those on the earth's surface should therefore have no discernible effect on their relative rates. Time dilation is a strictly asymmetric phenomenon according to both the Hafele-Keating experimental data [1, 2] and the everyday experience of the GPS methodology. The only rational conclusion that can be drawn under these circumstances is that *the LT is contradicted by these timing results*.

Moreover, there is another way that the clock-rate relationships contradict the LT. The assumption of strict proportionality between the rates of clocks on the satellite and those on the earth's surface is only consistent with the conclusion that the remote non-simultaneity of events predicted by the LT also does not occur in practice. Assume, for example, that the satellite clocks run slower by a constant factor Q>0 than those on the ground. That means if two events are simultaneous for the ground observer ($T_1=T_2$), they must also be simultaneous for the satellite observer. His corresponding times will both be shorter than for the ground observer, but they will also be equal to each other ($Q^{-1}T_1=Q^{-1}T_2$) because of the proportionality relationship between their respective clock rates.

3. Simultaneity and Causality: A GPS-compatible Lorentz Transformation:

The question that arises from the above considerations is whether relativity theory, unlike the LT version, can be formulated so that the experimentally verified findings of asymmetric time dilation and remote simultaneity of events are predicted from it in a thoroughly straightforward manner. To investigate this possibility it is useful to go back and give closer consideration to the time relationship of the LT given in eq. (1). If the location x of the event changes while maintaining the same time t, it can be seen that the ratio t'/t must also change:

$$t'/t = \gamma\,(1 - vc^{-2}x/t). \qquad (3)$$

The latter quantity is the ratio of the measured times of the two proper clocks in the S and S' rest frames. If one assumes remote non-simultaneity of events, the fact that t/t' depends on the location of the event is of no consequence, but the evidence from the GPS methodology indicates strongly that all events do occur simultaneously for stationary observers in S and S'. If one assumes instead that remote simultaneity does hold, it is clear that t/t' is also the ratio of clock rates in the two rest frames. One is then forced to conclude that the rate of at least one of the clocks varies with the location of the event. This is a clear violation of causality.

There is another key conclusion to be drawn from eq. (1), however. In order to avoid a violation of causality, it is clearly necessary that the t/t' ratio be completely independent of the event's location. In short, the concept of space-time mixing that has become a staple of modern cosmology, as for example in string theory [9], must be eliminated in order to develop a theory of relativity that is consistent with experimental data for time dilation. Moreover, it is essential that the revised theory satisfy both of Einstein's postulates of relativity, because each of them has also been confirmed in numerous experimental investigations.

Fortunately, the above conditions are easily fulfilled. The reason is that there is a degree of freedom [10] in the definition of the Lorentz transformation. Lorentz noted as early as 1899 [11,12] that the LT is only defined within a normalization factor. If each of the right-hand sides of the LT is multiplied by the same factor, a new transformation is formed which leaves Maxwell's equations invariant. This procedure also guarantees that the light-speed postulate continues to be satisfied. This is evident because the velocity of light is computed by forming the ratio of each spatial coordinate with the corresponding time coordinate, in which case the normalization factor is simply cancelled out.

According to the above arguments based on asymmetric time dilation and remote simultaneity of events, there must be a strictly proportional relationship between the rates of clocks in any two inertial systems. To be concrete, one must replace eq. (2) of the LT by the simple proportionality relation given below:

$$\Delta t' = \Delta t/Q, \qquad (4)$$

where Q is a constant which is solely determined by the relationship between the two inertial systems S and S'. The original four equations of the LT are given below:

$$\Delta t' = \gamma\,(\Delta t - v\Delta x c^{-2}) = \gamma\,\eta^{-1}\,\Delta t \qquad (5a)$$
$$\Delta x' = \gamma\,(\Delta x - v\Delta t) \qquad (5b)$$
$$\Delta y' = \Delta y \qquad (5c)$$
$$\Delta z' = \Delta z. \qquad (5d)$$

In order to obtain the GPS-compatible version of the relativistic space-time transformation, including eq. (4), it is merely necessary to multiply each of the right-hand sides of these equations with the normalization factor $\eta(\gamma Q)^{-1}$. The resulting transformation is given below:

$$\Delta t' = \Delta t/Q \qquad (6a)$$
$$\Delta x' = (\eta/Q)\,(\Delta x - v\Delta t) \qquad (6b)$$
$$\Delta y' = \eta\Delta y/\gamma Q \qquad (6c)$$
$$\Delta z' = \eta\Delta z/\gamma Q. \qquad (6d)$$

In the following these equations will be referred to as the GPS-LT. They guarantee the remote simultaneity of events and asymmetric time dilation as well as to insure that there is no possible violation of causality, all of which relationships are confirmed by experimental data obtained in connection with the GPS methodology. They also satisfy Einstein's two postulates of relativity and are consistent with the same relativistic velocity transformation (RVT) as obtained in his original work [1], namely

$$u_x' = (1 - vu_xc^{-2})^{-1}(u_x - v) = \eta\,(u_x - v) \qquad (7a)$$
$$u_y' = \gamma^{-1}\,(1 - vu_xc^{-2})^{-1}\,u_y = \eta\,\gamma^{-1}\,u_y \qquad (7b)$$
$$u_z' = \gamma^{-1}\,(1 - vu_xc^{-2})^{-1}\,u_z = \eta\,\gamma^{-1}\,u_z, \qquad (7c)$$

where $u_x= \Delta x/\Delta t$ etc. are the velocity components. A number of the most important results of relativity theory actually result directly from the RVT, and thus do not rely in any way on Einstein's assumption about the normalization factor used to obtain the LT. These include the aberration of starlight at the zenith [13] and the Fresnel light-drag experiment [14], both of which were quite important in Einstein's thought process [15]. The RVT also guarantees compliance with the light-speed postulate. It is used directly in the derivation of the Thomas precession of a spinning electron [16,17] and thus the LT is not essential in this case either.

More details about the GPS-LT may be found elsewhere [18], including a proof that it is also consistent with the relativity principle (RP). The above discussion demonstrates that space-time mixing is not essential to satisfy the RP. The direct proportionality assumed in eq. (6a) between the respective clock rates in S and S' is quite consistent with experimental findings, as already been stated, but it also seemingly conflicts with the conventional view that all inertial systems are equivalent and therefore indistinguishable [19]. Galileo's original arguments when he introduced the RP in 1632 shed considerable light on this issue. He used the example of passengers locked in the hold of a ship who were trying to determine whether they were still located at the dock or were moving on a perfectly calm sea [20]. His main point was that it would be impossible for them to make this determination on the basis of their purely in situ observations. More interesting in the present context, however, is that this argument does not exclude the possibility that objects on the ship, including the passengers themselves, did not undergo changes in their physical measurements as a result of the ship's motion. Rather, the assertion is that all such changes must be perfectly uniform, and that this is the fundamental reason why no distinction can be observed without carrying out measurements outside the ship's hold. The above interpretation is also consistent with Einstein's original work [1] in which he concluded that acceleration of a clock leads to a decrease in its rate. After the acceleration phase is concluded and a new state of motion is reached, it seems reasonable to assume that the clock's rate continues to be slower than in its original state. The RP simply states that the rates of all clocks are altered in the same proportion when they make the transition between the same two inertial systems. Similarly as with the First Law of Thermodynamics, it does not matter which intermediate states were reached in the process as long as the initial and final states are identical [21].

4. GPS Clock-rate Adjustments:

The rates of clocks are known experimentally to change with both their state of motion (time dilation) and their position in a gravitational field (red shift). The Hafele-Keating study [1,2] found that the earth's center of mass (ECM) plays a central role in each case. The fractional change in rate depends on the speed of a given clock relative to that position as well as the corresponding difference in gravitational potential. As a result, in comparing different clocks on the earth's surface, it is necessary to know both the latitude χ of each clock as well as its altitude h relative to sea level [7]. The slowing down of clock rates due to their motion is inversely proportional to γ ($R_E\Omega\cos\chi$) , where Ω is the earth's rotational frequency (2π radians per 24 h = 86 400 s) and R_E is the earth's radius (or more accurately, the distance between the location of the clock and the ECM).

In order to have a network of clocks located on the earth's surface, it is first necessary to designate one (Z) as a standard (note that in the following discussion it is assumed that all clocks run at constant rates). Theoretically, there is no restriction on its location. Its latitude χ_Z and altitude r_Z relative to the ECM are then important parameters in computing the ratio of the rates of each clock in the network with that of the standard clock. For this purpose it is helpful to define the ratio Q as follows:

$$Q= \gamma\ (R_E\Omega\cos\chi)/\ R_E\Omega\cos\chi_Z). \quad (8)$$

This ratio tells us how much slower (if Q>1) or faster (if Q<1) the given (secondary) clock runs than the standard if both are located at the same gravitational potential. The gravitational red shift needs to be taken into account to obtain the actual clock-rate ratio, however. For this purpose, it is helpful to define a second ratio S for each secondary clock:

$$S = 1 + g(r-r_Z)c^{-2}, \quad (9)$$

where r is the distance of the clock to the ECM. This ratio tells us how much faster (S>1) the clock runs relative to the standard by virtue of their difference in gravitational potential. The elapsed time Δt on the clock for a given event can then be converted to the corresponding elapsed time Δt_Z on the standard clock by combining the two ratios as follows

$$\Delta t_Z = QS^{-1}\Delta t. \quad (10)$$

It is possible to obtain the above ratios without having any communication between the laboratories that house the respective clocks. The necessary synchronization can begin by sending a light signal directly from the position of a second clock A lying closest to Z. The corresponding distance can be determined to as high an accuracy as possible using GPS. Division by c then gives the elapsed time read from Z for the one-way travel of the signal. The time of arrival on the standard clock is then adjusted backward by this amount to give the time of emission $T^{S0}(Z)$ for the signal, again as read from clock Z. The corresponding time of the initial emission read from clock A is also stored with the value $T^0(A)$. In principle, all subsequent timings can be determined by subtracting $T^0(A)$ from the current reading on clock A to obtain Δt to be inserted in eq. (10). The time T^Z of the event on the standard clock is then computed to be:

$$T^Z = QS^{-1} \Delta t + T^{S0}(A), \qquad (11)$$

where Q and S are the specific values of the ratios computed above for clock A.

Once the above procedure has been applied to clock A, it attains equivalent status as a standard. The next step therefore can be applied to the clock which is nearest either to clock A or clock Z. In this way the network of standard clocks can be extended indefinitely across the globe. Making use of the "secondary" standard (A) naturally implies that all timings there are based on its adjusted readings. It is important to understand that no physical adjustments need to be made to the secondary clock, rather its direct readings are simply combined with the Q and S factors in eq. (11) to obtain the timing results for a hypothetical standard. A discussion of this general point has been given earlier by van Flandern [6]. The situation is entirely analogous to having a clock in one's household that runs systematically slower than the standard rate. One can nonetheless obtain accurate timings by multiplying the readings from the faulty clock by an appropriate factor and keeping track of the time that has elapsed since it was last set to the correct time. The key word in this discussion is "systematic." If the error is always of quantitatively reliable magnitude, the faulty clock can replace the standard without making any repairs.

The same principles used to standardize clock rates on the earth's surface can also be applied for adjusting GPS satellite clocks. More details about such procedures may be found elsewhere [22,23], so only a brief summary will be given in the present work. Assume that the clock is running at the standard rate prior to launch and is perfectly synchronized with the standard clock (i.e. as adjusted at the local position). In order to illustrate the principles involved, the gravitational effects of other objects in the neighborhood of the satellite are neglected in the following discussion, as well as inhomogeneous characteristics of the Earth's gravitational field. The main difference relative to the previous example is that the Q and S factors needed to make the adjustment from local to standard clock rate using eqs. (10,11) are no longer constant. Their computation requires a precise knowledge of the trajectory of the satellite, specifically the current value of its speed v and altitude r relative to the ECM. The acceleration due to gravity changes in flight and so the ratio S also has to be computed in a more fundamental way. For this purpose, it is helpful to define the following quantity connected with the gravitational potential:

$$A\,(r) = 1 + GM_E/c^2r, \qquad (12)$$

where M_E is the gravitational mass of the earth (5.975×10^{24} kg) and G is Newton's Universal Gravitation Constant (6.67×10^{-11} Nm^2/kg^2). The value of S is therefore given as the ratio of the A (r) values for the satellite and the standard clock:

$$S = A\,(r_Z)/\,A\,(r), \qquad (13)$$

which simplifies to eq. (9) near the earth's surface (with $g = GM_E r_Z^{-2}$). The corresponding value of the Q ratio is at least simple in form:

$$Q = \gamma\,(v)/\gamma\,(v_Z) = \gamma\,(v)/\gamma\,(R_E \Omega \cos \chi_Z). \qquad (14)$$

Note that the latitude χ_Z is not that of the launch position relative to the ECM, but rather that of the original standard clock. The accuracy of the adjustment procedure depends primarily on the determination of the satellite speed v relative to the ECM at each instant.

In this application the underlying principle is to adjust the satellite clock rate to the corresponding standard value over the entire flight, including the period after orbit has been achieved [22]. The correction is made continuously in small intervals by using eq. (10) and the current values of Q and S in each step. The result is tantamount to having the standard clock running at its normal rate on the satellite. This above procedure supercedes the "pre-correction" technique commonly discussed in the literature [7] according to which the satellite clock is *physically* adjusted prior to launch. The latter's goal is to approximately correct for the estimated change in clock rate expected if the satellite ultimately travels in a constant circular trajectory once it achieves orbital speed. The present theoretical procedure has the advantage of being able to account for departures from a perfectly circular orbit and also for rate changes occurring during the launch phase.

5. Conclusion:

The experiments carried out with atomic clocks carried onboard circumnavigating airplanes [1,2] indicate that time dilation is fundamentally *asymmetric*. These timing results show quantitatively that clock rates in different rest frames are always strictly proportional to one another. The GPS methodology assumes such a relationship between atomic clocks located on satellites and their identical counterparts of the earth's surface in order to ensure accurate distance measurements. This experience stands in direct contradiction to the prediction of the Lorentz transformation (LT) of *symmetric time dilation*, i.e. where a clock in motion always appears to run more slowly than one that is stationary in the rest frame of the observer. Clock-rate proportionality also precludes the existence of remote non-simultaneity, because events which occur simultaneously for a proper clock on a satellite must also occur at the same time for clocks on earth running at a strictly proportional rate. The popular concept of the inextricable mixing of space and time is also negated on the same basis.

Relativity theory can be amended to account for clock-rate proportionality while still satisfying Einstein's two postulates. One simply demands that the ratios of measured elapsed times Δt and $\Delta t'$ for the same event for observers in different inertial rest frames S and S' always have the same constant value, i.e. $\Delta t'=\Delta t/Q$ in eq. (4). There is a degree of freedom in the most general form of the Lorentz transformation (GLT) in the form of a normalization constant which can be uniquely determined on this basis. The result is an alternative space-time transformation referred to as the GPS-LT. It is compatible with the same velocity transformation (RVT) introduced by Einstein in his original 1905 paper, but unlike the LT, it is consistent with asymmetric time dilation and is quantitatively verified in all relevant timing experiments carried out to the present time. It belies the conventional wisdom that space and time must be bound in a single four-dimensional entity in order to satisfy the light-speed postulate. The GPS-LT is also applicable, on an instantaneous basis, for rest frames which are subject to unbalanced external forces, such as were present in the Hafele-Keating and Hay et al. experiments.

The parameter Q in the GPS-LT can be looked upon as a conversion factor between the units of time in two different rest frames. It can easily be computed if the speeds of the two rest frames are known relative to a specific reference. It is the ECM in the case of the atomic clocks employed in the experiments with circumnavigating airplanes and the axis of the rotor in the Hay et al. transverse Doppler effect study. There is a corresponding conversion factor S which takes account of gravitational effects on clock rates. It also can be easily computed if the respective positions in a gravitational field are known for the two rest frames of interest. Knowledge of both the Q and S factors on a continuous basis can be used to set up a network of atomic clocks on the earth's surface. Atomic clocks onboard GPS satellites can also be adjusted numerically with analogous techniques to ensure that they run at the same rate as an identical standard clock located on the earth's surface. This procedure promises to be more accurate than the "pre-correction" method conventionally used for this purpose in the GPS technology. It has the potential of accurately and instantaneously accounting for changes in the rates of proper satellite clocks without making actual physical adjustments prior to launch.

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