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SOME IDENTITIES OF OVERPARTITION PAIRS INTO ODD PARTS

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Abstract: Recently, Bernard L.S. Lin has studied various arithmetic properties of the function $\overline{pp_0}(n)$, the number of overpartition pairs of n into odd parts. In particular, he has obtained a number of Ramanujan-type congruences modulo 3 and modulo powers of 2. In this paper, we give proof of some of these congruences and find some other interesting congruences by employing elementary generating function dissection techniques.

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1. Introduction:

An overpartition of a positive integer n is a non increasing sequence of positive integers whose sum is n in which first occurrence of a number may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n and $\overline{p_0}(n)$ denote the number of overpartitions of n in which only odd parts are used. For example, the overpartitions of 3 are

$$3, \quad \overline{3}, \quad 2+1, \quad \overline{2}+1, \quad 2+\overline{1}, \quad \overline{2}+\overline{1}, \quad 1+1+1, \quad \overline{1}+1+1.$$

Thus, from this example, $\overline{p}(3) = 8$ and $\overline{p_0}(3) = 4$.

The function $\overline{p}(n)$ has been considered recently by number of mathematicians including Corteel and Lovejoy [7], Hirschhorn and Sellers [8, 9], Mahlburg [14] and Kim [11]. In [8] and [14], several Ramanujan-like congruences modulo small powers of 2 are proved for $\overline{p}(n)$. In [10], Hirschhorn and Sellers found several interesting results for $\overline{p}_0(n)$ including Ramanujan-type congruences modulo powers of 2.

Recently, arithmetic properties of overpartition pairs $\overline{pp}(n)$ have been considered by Bringmann and Lovejoy [5], Chen and Lin [6] and Kim [12]. In [10] Hirschhorn and Sellers studied the arithmetic properties of overpartitions using only odd parts. More recently, Lin [13] has investigated various arithmetic properties of overpartition pairs into odd parts. He has obtained a number of Ramanujan type congruences modulo 3 and modulo powers of 2. An overpartition pairs into odd parts is a pair of overpartitions (λ, μ) such that the parts of both overpartitions λ and μ are restricted to be odd integers. Note that either λ or μ may be an overpartition of zero. Let $\overline{pp}_0(n)$ denote the number of overpartition pairs of n into odd parts. Then the generating function for $\overline{pp}_0(n)$ is

$$\sum_{n=0}^{\infty} \overline{pp}_0(n) q^n = \frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^4 (q^4;q^4)_\infty^2}. \tag{1}$$

where, here and the sequel, for $|q| < 1$ and positive integers n, we use the standard notation

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \quad \text{and} \quad (a;q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n).$$

We list our main results in the following six theorems.

Theorem 1.1 [13, Theorem] We have

$$\sum_{n=0}^{\infty}\overline{pp}_0(2n)q^n=\frac{(q^2;q^2)_\infty^{12}}{(q;q)_\infty^8(q^4;q^4)_\infty^4}, \tag{2}$$
$$\sum_{n=0}^{\infty}\overline{pp}_0(2n+1)q^n=4\frac{(q^4;q^4)_\infty^4}{(q;q)_\infty^4}. \tag{3}$$

Theorem 1.2. We have

$$\sum_{n=0}^{\infty}\overline{pp}_0(4n)q^n=4\frac{(q^2;q^2)_\infty^{24}}{(q;q)_\infty^{16}(q^4;q^4)_\infty^8}+16q\,\frac{(q^4;q^4)_\infty^8}{(q;q)_\infty^8}, \tag{4}$$
$$\sum_{n=0}^{\infty}\overline{pp}_0(4n+1)q^n=4\frac{(q^2;q^2)_\infty^{18}}{(q;q)_\infty^{14}(q^4;q^4)_\infty^4}, \tag{5}$$
$$\sum_{n=0}^{\infty}\overline{pp}_0(4n+2)q^n=8\frac{(q^2;q^2)_\infty^{12}}{(q;q)_\infty^{12}}, \tag{6}$$
$$\sum_{n=0}^{\infty}\overline{pp}_0(4n+3)q^n=16\frac{(q^2;q^2)_\infty^{6}(q^4;q^4)_\infty^4}{(q;q)_\infty^{10}}. \tag{7}$$

Lin [13] has proved some of the identities given in Theorem 1.1 and Theorem 1.2.

Theorem 1.3. We have

$$\sum_{n=0}^{\infty}\overline{pp}_0(8n+4)q^n=80\times\left\{\frac{(q^2;q^2)_\infty^{36}}{(q;q)_\infty^{28}(q^4;q^4)_\infty^8}+16q\,\frac{(q^2;q^2)_\infty^{12}(q^4;q^4)_\infty^8}{(q;q)_\infty^{20}}\right\}, \tag{8}$$
$$\sum_{n=0}^{\infty}\overline{pp}_0(8n+6)q^n=32\times\left\{3\frac{(q^2;q^2)_\infty^{30}}{(q;q)_\infty^{26}(q^4;q^4)_\infty^4}+16q\,\frac{(q^2;q^2)_\infty^{6}(q^4;q^4)_\infty^{12}}{(q;q)_\infty^{8}}\right\}, \tag{9}$$
$$\sum_{n=0}^{\infty}\overline{pp}_0(8n+7)q^n=32\times\left\{5\frac{(q^4;q^4)_\infty^{19}(q^2;q^2)_\infty^6}{(q;q)_\infty^{19}(q^8;q^8)_\infty^6}+40q\,\frac{(q^2;q^2)_\infty^{10}(q^4;q^4)_\infty^7(q^8;q^8)_\infty^2}{(q;q)_\infty^{19}}\right.$$
$$\left.+16q^2\,\frac{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^{10}}{(q;q)_\infty^{19}(q^4;q^4)_\infty^5}\right\}. \tag{10}$$

Theorem 1.4. We have

$$\sum_{n=0}^{\infty}\overline{pp}_0(12n+6)q^n=24\left\{4\frac{(q^2;q^2)_\infty^{19}(q^3;q^3)_\infty^{29}}{(q;q)_\infty^{35}(q^6;q^6)_\infty^{13}}+363q\,\frac{(q^2;q^2)_\infty^{16}(q^3;q^3)_\infty^{20}}{(q;q)_\infty^{32}(q^6;q^6)_\infty^{4}}\right.$$
$$\left.+2496q^2\,\frac{(q^2;q^2)_\infty^{13}(q^3;q^3)_\infty^{11}(q^6;q^6)_\infty^5}{(q;q)_\infty^{29}}+1408q^3\,\frac{(q^2;q^2)_\infty^{10}(q^3;q^3)_\infty^{2}(q^6;q^6)_\infty^{14}}{(q;q)_\infty^{26}}\right\}, \tag{11}$$

$$\sum_{n=0}^{\infty}\overline{pp}_0(12n+10)q^n=48\left\{13\frac{(q^2;q^2)_\infty^{18}(q^3;q^3)_\infty^{26}}{(q;q)_\infty^{34}(q^6;q^6)_\infty^{10}}+444q\,\frac{(q^2;q^2)_\infty^{15}(q^3;q^3)_\infty^{17}}{(q;q)_\infty^{21}(q^6;q^6)_\infty}\right.$$
$$\left.+1416q^2\,\frac{(q^2;q^2)_\infty^{12}(q^3;q^3)_\infty^{8}(q^6;q^6)_\infty^8}{(q;q)_\infty^{28}}+256q^3\,\frac{(q^2;q^2)_\infty^{9}(q^6;q^6)_\infty^{7}}{(q;q)_\infty^{25}(q^3;q^3)_\infty}\right\} \tag{12}$$

From the above identities, we easily deduce the following congruences.

Theorem 1.5. We have

$$\begin{aligned}
\overline{pp}_0(2n+1) &\equiv 0 \pmod 4,\\
\overline{pp}_0(4n) &\equiv 0 \pmod 4,\\
\overline{pp}_0(4n+1) &\equiv 0 \pmod 4,\\
\overline{pp}_0(4n+2) &\equiv 0 \pmod 8,\\
\overline{pp}_0(4n+3) &\equiv 0 \pmod{16},\\
\overline{pp}_0(8n+4) &\equiv 0 \pmod{80},\\
\overline{pp}_0(8n+6) &\equiv 0 \pmod{32},\\
\overline{pp}_0(8n+7) &\equiv 0 \pmod{32},\\
\overline{pp}_0(12n+6) &\equiv 0 \pmod{24},\\
\overline{pp}_0(12n+10) &\equiv 0 \pmod{48}. \qquad (13)
\end{aligned}$$

We also prove some new congruences modulo powers of 2 given in the following theorem. Theorem 1.6. We have

$$\overline{pp}_0(8n+3) \equiv 0 \pmod{2^4}, \qquad (14)$$
$$\overline{pp}_0(8n+5) \equiv 0 \pmod{2^4}, \qquad (15)$$
$$\overline{pp}_0(8n+6) \equiv 0 \pmod{2^5}, \qquad (16)$$
$$\overline{pp}_0(8n+7) \equiv 0 \pmod{2^5}, \qquad (17)$$
$$\overline{pp}_0(16n+6) \equiv 0 \pmod{2^5}, \qquad (18)$$
$$\overline{pp}_0(16n+8) \equiv 0 \pmod{2^4}, \qquad (19)$$
$$\overline{pp}_0(16n+10) \equiv 0 \pmod{2^5}, \qquad (20)$$
$$\overline{pp}_0(16n+12) \equiv 0 \pmod{2^4}, \qquad (21)$$
$$\overline{pp}_0(16n+14) \equiv 0 \pmod{2^5}, \qquad (22)$$
$$\overline{pp}_0(32n+20) \equiv 0 \pmod{320}, \qquad (23)$$
$$\overline{pp}_0(32n+28) \equiv 0 \pmod{320}, \qquad (24)$$
$$\overline{pp}_0(48n+2) \equiv 0 \pmod{2^3}, \qquad (25)$$
$$\overline{pp}_0(48n+10) \equiv 0 \pmod{2^5}, \qquad (26)$$
$$\overline{pp}_0(48n+18) \equiv 0 \pmod{2^3}, \qquad (27)$$
$$\overline{pp}_0(48n+26) \equiv 0 \pmod{2^5}, \qquad (28)$$
$$\overline{pp}_0(48n+34) \equiv 0 \pmod{2^5}, \qquad (29)$$
$$\overline{pp}_0(48n+42) \equiv 0 \pmod{2^5}. \qquad (30)$$

2. Some lemmas:

In order to prove the above identities and congruences we first give some lemmas.

Lemma 2.1. We have

$$\frac{1}{(q;q)_\infty^2} = \frac{(q^8;q^8)_\infty^5}{(q^2;q^2)_\infty^5(q^{16};q^{16})_\infty^2} + 2q\frac{(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^5(q^8;q^8)_\infty}, \tag{31}$$

$$\frac{1}{(q;q)_\infty^4} = \frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4} + 4q\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}, \tag{32}$$

$$\frac{1}{(q;q)_\infty^8} = \frac{(q^4;q^4)_\infty^{28}}{(q^2;q^2)_\infty^{28}(q^8;q^8)_\infty^8} + 8q\frac{(q^4;q^4)_\infty^{16}}{(q^2;q^2)_\infty^{24}} + 16q^2\frac{(q^4;q^4)_\infty^4(q^8;q^8)_\infty^8}{(q^2;q^2)_\infty^{20}}, \tag{33}$$

$$\begin{aligned}\frac{1}{(q;q)_\infty^{16}} &= \frac{(q^4;q^4)_\infty^{56}}{(q^2;q^2)_\infty^{56}(q^8;q^8)_\infty^{16}} + 16q\frac{(q^4;q^4)_\infty^{44}}{(q^2;q^2)_\infty^{52}(q^8;q^8)_\infty^8} + 96q^2\frac{(q^4;q^4)_\infty^{32}}{(q^2;q^2)_\infty^{48}}\\ &\quad + 256q^3\frac{(q^4;q^4)_\infty^{20}(q^8;q^8)_\infty^8}{(q^2;q^2)_\infty^{44}} + 256q^4\frac{(q^4;q^4)_\infty^8(q^8;q^8)_\infty^{16}}{(q^2;q^2)_\infty^{40}},\end{aligned} \tag{34}$$

$$\begin{aligned}\frac{1}{(q;q)_\infty^{12}} &= \frac{(q^4;q^4)_\infty^{42}}{(q^2;q^2)_\infty^{42}(q^8;q^8)_\infty^{12}} + 12q\frac{(q^4;q^4)_\infty^{30}}{(q^2;q^2)_\infty^{38}(q^8;q^8)_\infty^4} + 48q^2\frac{(q^4;q^4)_\infty^{18}(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{34}}\\ &\quad + 64q^3\frac{(q^4;q^4)_\infty^6(q^8;q^8)_\infty^{12}}{(q^2;q^2)_\infty^{30}}.\end{aligned} \tag{35}$$

Proof. Adding identities (v) and (vi) of [1, Entry 25, p. 40], we have

$$\varphi(q) = \varphi(q^4) + 2q\psi^2(q^8). \tag{36}$$

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4). \tag{37}$$

$$\varphi^4(q) = \varphi^4(q^2) + 8q\varphi^2(q^2)\psi^2(q^4) + 16q^2\psi^4(q^4). \tag{38}$$

where

$$\varphi(q) := f(q,q) = \sum_{k=-\infty}^{\infty} q^{k^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}. \tag{39}$$

and

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{40}$$

Now, employing (39) and (40) in (36), (54), (38), we readily arrive at (31), (32) and (33).

Again replacing q by $-q$ in (39), we have

$$\varphi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)}. \tag{41}$$

After Ramanujan, we also define

$$\chi(-q) = (q;q^2)_\infty = \frac{(q;q)_\infty}{(q^2;q^2)_\infty}. \tag{42}$$

Lemma 2.2. We have

$$(q;q)_\infty^3 = \frac{(q^6;q^6)_\infty(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty(q^{18};q^{18})_\infty^3} - 3q(q^9;q^9)_\infty^3 + 4q^3\frac{(q^3;q^3)_\infty^2(q^{18};q^{18})_\infty^6}{(q^6;q^6)_\infty^2(q^9;q^9)_\infty^3}. \tag{43}$$

Lemma 2.3. We have

$$\frac{(q^4;q^4)_\infty}{(q;q)_\infty} = \frac{(q^{12};q^{12})_\infty (q^{18};q^{18})_\infty^4}{(q^3;q^3)_\infty^3 (q^{36};q^{36})_\infty^2} + q\,\frac{(q^6;q^6)_\infty^2 (q^9;q^9)_\infty^3 (q^{36};q^{36})_\infty}{(q^3;q^3)_\infty^4 (q^{18};q^{18})_\infty^2} + 2q^2\,\frac{(q^6;q^6)_\infty (q^{18};q^{18})_\infty (q^{36};q^{36})_\infty}{(q^3;q^3)_\infty^3}. \tag{44}$$

Proof. From [2], we have

$$\frac{c(q)}{c(q^4)} = 1 + \frac{\psi^2(q^2)}{q\psi^2(q^6)}. \tag{45}$$

Next, we recall from [4] that

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} = 3q^{1/3}\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}. \tag{46}$$

Employing (46) in (45), we find that

$$\frac{(q^4;q^4)_\infty}{(q;q)_\infty} = q\frac{(q^{12};q^{12})_\infty^3}{(q^3;q^3)^3}\left\{1 + \frac{\psi^2(q^2)}{q\psi^2(q^6)}\right\}. \tag{47}$$

Next, replacing q by q^2 in (56), we have

$$\psi^2(q^2) = \frac{\varphi^2(-q^{18})}{\chi^2(-q^6)} + q^4\psi^2(q^{18}) + 2q^2\frac{\varphi(-q^{18})\psi(q^{18})}{\chi(-q^6)}. \tag{48}$$

Using (48) in (47), we obtain

$$\begin{aligned}\frac{(q^4;q^4)_\infty}{(q;q)_\infty} &= q\frac{(q^{12};q^{12})_\infty^3}{(q^3;q^3)_\infty^3}\left\{1 + \frac{1}{q\psi^2(q^6)}\left(\frac{\varphi^2(-q^{18})}{\chi^2(-q^6)} + q^4\psi^2(q^{18}) + 2q^2\frac{\varphi(-q^{18})\psi(q^{18})}{\chi(-q^6)}\right)\right\}\\ &= q\frac{(q^{12};q^{12})_\infty^3}{(q^3;q^3)_\infty^3}\left\{1 + q^3\frac{\psi^2(q^{18})}{\psi^2(q^6)}\right\} + q\frac{(q^{12};q^{12})_\infty^3}{(q^3;q^3)_\infty^3}\left\{\frac{\varphi^2(-q^{18})}{q\psi^2(q^6)\chi^2(-q^6)} + 2q\,\frac{\varphi(-q^{18})\psi(q^{18})}{\psi^2(q^6)\chi(-q^6)}\right\}.\end{aligned} \tag{49}$$

Employing (40), (41) and (42) in (49), we find that

$$\frac{(q^4;q^4)_\infty}{(q;q)_\infty} = q\frac{(q^{12};q^{12})_\infty^3}{(q^3;q^3)_\infty^3}\left\{1 + q^3\frac{\psi^2(q^{18})}{\psi^2(q^6)}\right\} + \frac{(q^{12};q^{12})_\infty (q^{18};q^{18})_\infty^4}{(q^3;q^3)_\infty^3 (q^{36};q^{36})_\infty^2} + 2q^2\,\frac{(q^{18};q^{18})_\infty (q^{36};q^{36})_\infty (q^6;q^6)_\infty}{(q^3;q^3)_\infty^3}. \tag{50}$$

Now, multiplying both sides of (47) by $\psi^2(q^6)/\psi^2(q^2)$, replacing q by q^3, and then employing (40), we deduce that

$$1 + q^3\,\frac{\psi^2(q^{18})}{\psi^2(q^6)} = \frac{(q^6;q^6)_\infty^2 (q^9;q^9)_\infty^3 (q^{36};q^{36})_\infty}{(q^3;q^3)_\infty (q^{18};q^{18})_\infty^2 (q^{12};q^{12})_\infty^3}. \tag{51}$$

Employing (51) in (50), we arrive at (50) to finish the proof.

Now we state a lemma.

Lemma 2.4. We have

$$\psi(q) = f(q^3,q^6) + q\psi(q^9), \tag{52}$$

$$f(q,q^2) = \frac{\varphi(-q^3)}{\chi(-q)}, \tag{53}$$

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4). \tag{54}$$

Proof. See [1, p. 49, Corollary(ii) and [1, p. 350, Eq. (2.3)] for the proofs of (52) and (53), respectively. Adding identities (v) and (vi) of [1, Entry 25, p. 40], we can easily derive (54).

Lemma 2.5. We have

$$\frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^6} = \frac{(q^6;q^6)_\infty^{10}(q^9;q^9)_\infty^{16}}{(q^3;q^3)_\infty^{18}(q^{18};q^{18})_\infty^8} + 6q\,\frac{(q^6;q^6)_\infty^{9}(q^9;q^9)_\infty^{13}}{(q^3;q^3)_\infty^{17}(q^{18};q^{18})_\infty^5} + 21q^2\,\frac{(q^6;q^6)_\infty^{8}(q^9;q^9)_\infty^{10}}{(q^3;q^3)_\infty^{16}(q^{18};q^{18})_\infty^2}$$
$$+ 44q^3\,\frac{(q^6;q^6)_\infty^{7}(q^9;q^9)_\infty^{7}(q^{18};q^{18})_\infty}{(q^3;q^3)_\infty^{15}} + 60q^4\,\frac{(q^6;q^6)_\infty^{6}(q^9;q^9)_\infty^{4}(q^{18};q^{18})_\infty^4}{(q^3;q^3)_\infty^{14}}$$
$$+ 48q^5\,\frac{(q^6;q^6)_\infty^{5}(q^9;q^9)_\infty(q^{18};q^{18})_\infty^7}{(q^3;q^3)_\infty^{13}} + 16q^6\,\frac{(q^6;q^6)_\infty^{4}(q^{18};q^{18})_\infty^{10}}{(q^9;q^9)_\infty^{2}(q^3;q^3)_\infty^{12}}. \tag{55}$$

Proof. Squaring both sides of (52) and then employing (53), we have

$$\psi^2(q) = \frac{\varphi^2(-q^9)}{\chi^2(-q^3)} + q^2\psi^2(q^9) + 2q\frac{\varphi(-q^9)\psi(q^9)}{\chi(-q^3)}. \tag{56}$$

We have from [3, Eq. 2.2, Theorem 2.1].

$$\frac{1}{\varphi(-q)} = \frac{\varphi^3(-q^9)}{\varphi^4(-q^3)} + 2q\frac{\varphi^3(-q^9)w(q^3)}{\varphi^4(-q^3)} + 4q^2w^2(q^3)\frac{\varphi^3(-q^9)}{\varphi^4(-q^3)}, \tag{57}$$

where

$$w(q) = \frac{(q;q)_\infty(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty(q^3;q^3)_\infty^3}. \tag{58}$$

Squaring both sides of (57), we find that

$$\frac{1}{\varphi^2(-q)} = \frac{\varphi^6(-q^9)}{\varphi^8(-q^3)}\{1 + 4qw(q^3) + 12q^2w^2(q^3) + 16q^3w^3(q^3) + 16q^4w^4(q^3)\}. \tag{59}$$

Multiplying (56) and (59) and then employing (40), (58), (41) and (42), we easily arrive at (55) to complete the proof.

3. Proofs of theorems :

Proof of Theorem 1.2. Employing (32) in (1), we have

$$\sum_{n=0}^{\infty} \overline{pp}_0(n)q^n = \frac{(q^2;q^2)_\infty^6}{(q^4;q^4)_\infty^2}\left\{\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4} + 4q\,\frac{(q^4;q^4)_\infty^{2}(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}\right\}. \tag{60}$$

Extracting from both sides of (60), those terms involving only q^{2n}, and then replacing q^2 by q, we arrive at (2). Again, extracting from both sides of (61), those terms involving only q^{2n+1}, and then replacing q^2 by q, we arrive at (3).

Proof of Theorem 1.2. Employing (33) in (2), we have

$$\sum_{n=0}^{\infty} \overline{pp}_0(2n)q^n = \frac{(q^2;q^2)_\infty^{12}}{(q^4;q^4)_\infty^4}\left\{\frac{(q^4;q^4)_\infty^{28}}{(q^2;q^2)_\infty^{28}(q^8;q^8)_\infty^8} + 8q\frac{(q^4;q^4)_\infty^{16}}{(q^2;q^2)_\infty^{24}} + 16q^2\,\frac{(q^4;q^4)_\infty^{4}(q^8;q^8)_\infty^8}{(q^2;q^2)_\infty^{20}}\right\}. \tag{61}$$

Extracting from both sides of (61), those terms involving only q^{2n}, and then replacing q^2 by q, we arrive at (4). Again, extracting from both sides of (61), those terms involving only q^{2n+1}, and then replacing q^2 by q, we arrive at (6).

Now, employing (32) in (3), we have

$$\sum_{n=0}^{\infty}\overline{pp}_0(2n+1)q^n = 4(q^4;q^4)_\infty^4\left\{\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4}+4q\,\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}\right\}. \tag{62}$$

Extracting from both sides of (62), those terms involving only q^{2n}, and then replacing q^2 by q, we arrive at (5). Again, extracting from both sides of (61), those terms involving only q^{2n+1}, and then replacing q^2 by q, we arrive at (7).

Proof of Theorem 1.3. Employing (33) and (34) in (4), we derive

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{pp}_0(4n)q^n &= 4\frac{(q^2;q^2)_\infty^{24}}{(q^4;q^4)_\infty^{8}}\Big\{\frac{(q^4;q^4)_\infty^{56}}{(q^2;q^2)_\infty^{56}(q^8;q^8)_\infty^{16}}+16q\frac{(q^4;q^4)_\infty^{44}}{(q^2;q^2)_\infty^{52}(q^8;q^8)_\infty^{8}}+96q^2\,\frac{(q^4;q^4)_\infty^{32}}{(q^2;q^2)_\infty^{48}}\\&\quad+256q^3\frac{(q^4;q^4)_\infty^{20}(q^8;q^8)_\infty^{8}}{(q^2;q^2)_\infty^{44}}+256q^4\frac{(q^4;q^4)_\infty^{8}(q^8;q^8)_\infty^{16}}{(q^2;q^2)_\infty^{40}}\Big\}\\&\quad+16q\,(q^4;q^4)_\infty^{8}\Big\{\frac{(q^4;q^4)_\infty^{28}}{(q^2;q^2)_\infty^{28}(q^8;q^8)_\infty^{8}}+8q\frac{(q^4;q^4)_\infty^{16}}{(q^2;q^2)_\infty^{24}}+16q^2\,\frac{(q^4;q^4)_\infty^{4}(q^8;q^8)_\infty^{8}}{(q^2;q^2)_\infty^{20}}\Big\}.\end{aligned} \tag{63}$$

Now, extracting from both sides of (63), those terms involving only q^{2n+1}, and then replacing q^2 by q, we arrive at (8).

Again, using (35) in (6), we find

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{pp}_0(4n+2)q^n &= 8(q^2;q^2)_\infty^{12}\Big\{\frac{(q^4;q^4)_\infty^{42}}{(q^2;q^2)_\infty^{42}(q^8;q^8)_\infty^{12}}+12q\frac{(q^4;q^4)_\infty^{30}}{(q^2;q^2)_\infty^{38}(q^8;q^8)_\infty^{4}}+48q^2\,\frac{(q^4;q^4)_\infty^{18}(q^8;q^8)_\infty^{4}}{(q^2;q^2)_\infty^{34}}\\&\quad+64q^3\frac{(q^4;q^4)_\infty^{6}(q^8;q^8)_\infty^{12}}{(q^2;q^2)_\infty^{30}}\Big\}.\end{aligned} \tag{64}$$

Now, extracting from both sides of (64), those terms involving only q^{2n+1}, and then replacing q^2 by q, we arrive at (9).

Proof of Theorem 1.4. Employing (55) in (6), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{pp}_0(4n+2)q^n &= 8\Big\{\frac{(q^6;q^6)_\infty^{20}(q^9;q^9)_\infty^{32}}{(q^3;q^3)_\infty^{36}(q^{18};q^{18})_\infty^{16}}+12q\,\frac{(q^6;q^6)_\infty^{19}(q^9;q^9)_\infty^{29}}{(q^3;q^3)_\infty^{35}(q^{18};q^{18})_\infty^{13}}+78q^2\,\frac{(q^6;q^6)_\infty^{18}(q^9;q^9)_\infty^{26}}{(q^3;q^3)_\infty^{34}(q^{18};q^{18})_\infty^{10}}\\&\quad+340q^3\,\frac{(q^6;q^6)_\infty^{17}(q^9;q^9)_\infty^{23}}{(q^3;q^3)_\infty^{33}(q^{18};q^{18})_\infty^{7}}+1089q^4\,\frac{(q^6;q^6)_\infty^{16}(q^9;q^9)_\infty^{20}}{(q^3;q^3)_\infty^{32}(q^{18};q^{18})_\infty^{4}}\\&\quad+2664q^5\,\frac{(q^6;q^6)_\infty^{15}(q^9;q^9)_\infty^{17}}{(q^3;q^3)_\infty^{21}(q^{18};q^{18})_\infty}+5064q^6\,\frac{(q^6;q^6)_\infty^{14}(q^9;q^9)_\infty^{14}(q^{18};q^{18})_\infty^{2}}{(q^3;q^3)_\infty^{30}}\\&\quad+7488q^7\,\frac{(q^6;q^6)_\infty^{13}(q^9;q^9)_\infty^{11}(q^{18};q^{18})_\infty^{5}}{(q^3;q^3)_\infty^{29}}+8496q^8\,\frac{(q^6;q^6)_\infty^{12}(q^9;q^9)_\infty^{8}(q^{18};q^{18})_\infty^{8}}{(q^3;q^3)_\infty^{28}}\\&\quad+7168q^9\,\frac{(q^6;q^6)_\infty^{11}(q^9;q^9)_\infty^{5}(q^{18};q^{18})_\infty^{11}}{(q^3;q^3)_\infty^{27}}+4224q^{10}\,\frac{(q^6;q^6)_\infty^{10}(q^9;q^3)_\infty^{2}(q^{18};q^{18})_\infty^{14}}{(q^3;q^3)_\infty^{26}}\\&\quad+1536q^{11}\,\frac{(q^6;q^6)_\infty^{9}(q^{18};q^{18})_\infty^{17}}{(q^3;q^3)_\infty^{25}(q^9;q^9)_\infty}+256q^{12}\,\frac{(q^6;q^6)_\infty^{8}(q^{18};q^{18})_\infty^{20}}{(q^3;q^3)_\infty^{24}(q^9;q^9)_\infty^{4}}\Big\}.\end{aligned} \tag{65}$$

Extracting from both sides of (65), those terms involving only q^{3n+1}, and then replacing q^3 by q, we arrive at (11). Again, extracting from both sides of (61), those terms involving only q^{3n+2}, and then replacing q^3 by q, we arrive at (12).

Proof of Theorem 1.6. From (3) we have

$$\sum_{n=0}^{\infty} \frac{\overline{pp}_0(2n+1)}{4} q^n = \frac{(q^4;q^4)_\infty^4}{(q;q)_\infty^4}.$$
$$\equiv (q^4;q^4)_\infty^3 \pmod 4, \tag{66}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{pp}_0(8n+3) q^n \equiv 0 \pmod{16}, \tag{67}$$
$$\sum_{n=0}^{\infty} \overline{pp}_0(8n+5) q^n \equiv 0 \pmod{16}, \tag{68}$$
$$\sum_{n=0}^{\infty} \overline{pp}_0(8n+7) q^n \equiv 0 \pmod{16}. \tag{69}$$

It follows from (67), (68) and (69) that (18), (20) and (22) holds.

We have from (5)

$$\sum_{n=0}^{\infty} \frac{\overline{pp}_0(4n+1)}{4} q^n = \frac{(q^2;q^2)_\infty^{18}}{(q;q)_\infty^{14}(q^4;q^4)_\infty^4}.$$
$$\equiv (q^2;q^2)_\infty^3 \pmod 2. \tag{70}$$

From above we can easily derive (14).

Again, we have from (6)

$$\sum_{n=0}^{\infty} \frac{\overline{pp}_0(4n+2)}{8} q^n = \frac{(q^2;q^2)_\infty^{12}}{(q;q)_\infty^{12}}.$$
$$\equiv \frac{(q^8;q^8)_\infty^3}{(q^4;q^4)_\infty^3} \pmod 4 \tag{71}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{pp}_0(8n+6) q^n \equiv 0 \pmod{32}, \tag{72}$$
$$\sum_{n=0}^{\infty} \overline{pp}_0(16n+6) q^n \equiv 0 \pmod{32}, \tag{73}$$
$$\sum_{n=0}^{\infty} \overline{pp}_0(16n+10) q^n \equiv 0 \pmod{32} \tag{74}$$

and

$$\sum_{n=0}^{\infty} \overline{pp}_0(16n+14) q^n \equiv 0 \pmod{32}. \tag{75}$$

It follows from (72), (73), (74) and (75) that (16), (18), (20) and (22) hold.

From (4) we have

$$\sum_{n=0}^{\infty} \frac{\overline{pp}_0(4n)}{4} q^n = \frac{(q^2;q^2)_\infty^{24}}{(q;q)_\infty^{16}(q^4;q^4)_\infty^8} + 4q\,\frac{(q^4;q^4)_\infty^8}{(q;q)_\infty^8}$$
$$\equiv \frac{(q^8;q^8)_\infty^6}{(q^4;q^4)_\infty^{12}} + 4q\,(q^4;q^4)_\infty^6 \pmod 4, \tag{76}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{pp}_0(16n+8)q^n \equiv 0 \pmod{16} \tag{77}$$

and

$$\sum_{n=0}^{\infty} \overline{pp}_0(16n+12)q^n \equiv 0 \pmod{16}. \tag{78}$$

It follows from (77) and (78) that (19) and (21) holds.

From (8) we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\overline{pp}_0(8n+4)}{80} q^n &= \left\{ \frac{(q^2;q^2)_\infty^{36}}{(q;q)_\infty^{28}(q^4;q^4)_\infty^{8}} + 16q\, \frac{(q^2;q^2)_\infty^{12}(q^4;q^4)_\infty^{8}}{(q;q)_\infty^{20}} \right\} \\ &\equiv \frac{(q^8;q^8)_\infty^{9}}{(q^4;q^4)_\infty^{15}} + 16q\, (q^4;q^4)_\infty^{3}(q^8;q^8)_\infty^{3} \pmod{4},\end{aligned} \tag{79}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{pp}_0(32n+20)q^n \equiv 0 \pmod{320} \tag{80}$$

and

$$\sum_{n=0}^{\infty} \overline{pp}_0(32n+28)q^n \equiv 0 \pmod{320}. \tag{81}$$

Now, (23) and (24) easily follows from (80) and (81).

Again, we have from (64)

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}_0(4n+2)q^n &= 8(q^2;q^2)_\infty^{12} \Big\{ \frac{(q^4;q^4)_\infty^{42}}{(q^2;q^2)_\infty^{42}(q^8;q^8)_\infty^{12}} + 12q \frac{(q^4;q^4)_\infty^{30}}{(q^2;q^2)_\infty^{38}(q^8;q^8)_\infty^{4}} + 48q^2\, \frac{(q^4;q^4)_\infty^{18}(q^8;q^8)_\infty^{4}}{(q^2;q^2)_\infty^{34}} \\ &+ 64q^3 \frac{(q^4;q^4)_\infty^{6}(q^8;q^8)_\infty^{12}}{(q^2;q^2)_\infty^{30}} \Big\},\end{aligned} \tag{82}$$

which yields that

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}_0(8n+2)q^n &\equiv 8 \frac{(q^2;q^2)_\infty^{42}}{(q;q)_\infty^{30}(q^4;q^4)_\infty^{12}} \pmod{32} \\ &\equiv 8\, (q^2;q^2)_\infty^{3} \pmod{32}.\end{aligned} \tag{83}$$

Employing (43) in (83), we find

$$\sum_{n=0}^{\infty} \overline{pp}_0(8n+2)q^n \equiv 8 \frac{(q^{12};q^{12})_\infty (q^{18};q^{18})_\infty^{6}}{(q^6;q^6)_\infty (q^{36};q^{36})_\infty^{3}} - 24q^2 (q^{18};q^{18})_\infty^{3} + 32q^6\, \frac{(q^6;q^6)_\infty^{2} (q^{36};q^{36})_\infty^{6}}{(q^{12};q^{12})_\infty^{2} (q^{18};q^{18})_\infty^{3}} \pmod{32}. \tag{84}$$

It follows from (84) that

$$\sum_{n=0}^{\infty} \overline{pp}_0(48n+2)q^n \equiv 0 \pmod 8, \tag{85}$$

$$\sum_{n=0}^{\infty} \overline{pp}_0(48n+10)q^n \equiv 0 \pmod{32}, \tag{86}$$

$$\sum_{n=0}^{\infty} \overline{pp}_0(48n+18)q^n \equiv 0 \pmod 8, \tag{87}$$

$$\sum_{n=0}^{\infty} \overline{pp}_0(48n+26)q^n \equiv 0 \pmod{32}, \tag{88}$$

$$\sum_{n=0}^{\infty} \overline{pp}_0(48n+34)q^n \equiv 0 \pmod{32}, \tag{89}$$

$$\sum_{n=0}^{\infty} \overline{pp}_0(48n+42)q^n \equiv 0 \pmod{32}. \tag{90}$$

Now, (25)–(30) are apparent from above.

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