RAMANUJAN’S CUBIC CONTINUED FRACTION AND RELATED RESULTS

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Abstract: On page 366 of his lost notebook [3], Ramanujan recorded a continued fraction known as Ramanujan’s Cubic Continued Fraction $G(q)$, defined as

$$G(q) = \frac{q^{\frac{1}{3}}}{1+}\ \frac{q+q^2}{1+}\ \frac{q^2+q^4}{1\ +\cdots} \qquad |q|<1$$

He also claimed that there are many results of $G(q)$ that are analogous to $R(q)$, Rogers-Ramanujan Continued Fraction. $G(q)$ and it’s other results are recorded in [2, 345-347] and [1, section 3.3] of his lost notebook. In this paper we derive some new identities involving Ramanujan’s Cubic continued fraction $G(q)$ by using Ramanujan’s theta function.

Keywords: Ramanujan’s cubic continued fraction; Theta functions

1. Introduction:

Throughout the paper, we assume $|q|<1$ and for positive integer n, we use the standard notation

$$(a;q)_0 := 1,\ (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \text{ and } \quad (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

The famous Rogers-Ramanujan continued fraction $R(q)$ is defined as

$$R(q) := \frac{q^{1/5}}{1+}\frac{q}{1+}\frac{q^2}{1+\ldots.} \qquad |q|<1 . \tag{1.1}$$

The continued fraction first appeared in a paper by L. J. Rogers in 1894. Ramanujan later rediscovered the Rogers-Ramanujan continued fraction and developed an extensive and deep theory for it. In his notebooks Ramanujan recorded many identities involving $R(q)$ which can be found in [Andrews, Berndt 2005; Berndt, 1991;Ramanujan, 1957; 1988]. One of the most important formulas for $R(q)$ is

$$\frac{1}{R(q)} - 1 - R(q) = \frac{(q^{1/5};q^{1/5})_\infty}{q^{1/5}(q^5;q^5)_\infty}. \tag{1.2}$$

Furthermore, on p. 206 of his lost notebook (Ramanujan, 1988), Ramanujan recorded the identities, namely,

$$\frac{1}{\sqrt{R(q)}} - \alpha\sqrt{R(q)} = \frac{1}{q^{1/10}}\sqrt{\frac{(q;q)_\infty}{(q^5;q\ )_\infty}}\prod_{n=1}^{\infty}\frac{1}{1+\alpha q^{n/5}+q^{2n/5}} \tag{1.3}$$

and

$$\frac{1}{\sqrt{R(q)}} - \beta\sqrt{R(q)} = \frac{1}{q^{1/10}}\sqrt{\frac{(q;q)_\infty}{(q^5;q\ )_\infty}}\prod_{n=1}^{\infty}\frac{1}{1+\beta q^{n/5}+q^{2n/5}} \tag{1.4}$$

where $\alpha = \dfrac{1-\sqrt{5}}{2}$ and $\beta = \dfrac{1+\sqrt{5}}{2}$.

Proof of these identities are given by Ramanathan (1984) and Berndt et.al (1999). See also [Andrews and Berndt, (2005), pp. 21-24].

On page 366 of his lost notebook , Ramanujan (1988) recorded a continued fraction known as Ramanujan's Cubic continued fraction $G(q)$, defined as

$$G(q) := \frac{q^{1/3}}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\ldots \quad |q|<1. \tag{1.5}$$

He also claimed that there are many results of $G(q)$ that are analogous to $R(q)$. It is known that

$$G(q) := q^{1/3}\frac{(q;q^2)_\infty}{(q^3;q^6)_\infty^3}. \tag{1.6}$$

$G(q)$ and it's other results are recorded in (Berndt, 1991, 345-347) and (Andrews and Berndt, 2005, section 3.3) of his lost notebook. Proofs of (1.7) can be found in papers by Selberg (1989), Gordon (1961), Andrews (1979).

In our work, we prove (1.10) by deriving product representations for $\frac{1}{\sqrt{G(q)}} \pm \sqrt{G(q)}$, namely

$$\frac{1}{\sqrt{G(q)}} + \sqrt{G(q)} = \frac{\sqrt{\chi^3(-q^3)}}{q^{1/6}\phi(-q^3)}.\sqrt{\chi(-q)}\psi(q^{1/3}), \tag{1.7}$$

$$\frac{1}{\sqrt{G(q)}} - 2\sqrt{G(q)} = \frac{\sqrt{\chi^3(-q^3)}}{q^{1/6}\phi(-q^3)}.\frac{\phi(-q^{1/3})}{\sqrt{\chi(-q)}}. \tag{1.8}$$

We also find some interesting identities related to $G(q)$. In the next section, we give definitions and preliminary results. In the last section, we present some new identities involving Ramanujan's cubic continued fraction $G(q)$.

2. Some preliminary definitions:

Ramanujan's general theta function is defined by

$$f(a,b) = \sum_{n=0}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab|<1 . \tag{2.1}$$

The function $f(a, b)$ satisfies the well-known Jacobi's triple product identity, which can be expressed as

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty . \tag{2.2}$$

Three special cases of $f(a, b)$ are defined, for $|q| < 1$, by (Berndt,1991, p.36, Entry 22)

$$\phi(q) = f(q,q) = \sum_{k=0}^{\infty} q^{k^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}, \tag{2.3}$$

$$\psi(q) = f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} == \frac{(q^2;q^2)^2{}_\infty}{(q;q)_\infty}, \tag{2.4}$$

$$f(-q) = f(-q,-q^2) = \sum_{k=0}^{\infty}(-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty}(-1)^k q^{k(3k+1)/2} = (q;q)_\infty. \tag{2.5}$$

where the product representations in (2.3)-(2.5) arise from (2.2).

Furthermore, the q-product representations of $\varphi(-q)$ and $\psi(-q)$ are given as

$$\varphi(-q) = \frac{(q;q)^2{}_\infty}{(q^2;q^2)_\infty} \quad \text{and} \quad \psi(-q) = \frac{(q;q)_\infty(q^4;q^4)_\infty}{(q^2;q^2)_\infty}. \tag{2.6}$$

We also define

$$\chi(q) = (-q;q^2)_\infty. \tag{2.7}$$

3. Main results and their proofs:

Theorem 3.1. We have

$$\frac{1}{\sqrt{G(q)}}+\sqrt{G(q)}=\frac{\sqrt{\chi^3(-q^3)}}{q^{1/6}\phi(-q^3)}.\sqrt{\chi(-q)}\psi(q^{1/3}), \tag{3.1}$$

$$\frac{1}{\sqrt{G(q)}}-2\sqrt{G(q)}=\frac{\sqrt{\chi^3(-q^3)}}{q^{1/6}\phi(-q^3)}.\frac{\phi(-q^{1/3})}{\sqrt{\chi(-q)}}. \tag{3.2}$$

Proof of (3.1). We recall from Berndt (1991), [p. 49,corollary(ii)], [p. 350, Eq.(2.3)] and [p. 39, Entry 24(iii)] that

$$\psi(q)= f(q^3,q^6)+q\psi(q^9), \tag{3.3}$$

$$\chi(-q)=\frac{\varphi(-q^3)}{f(q,q^2)}, \tag{3.4}$$

and

$$\chi^3(q)=\frac{\varphi(q)}{\psi(-q)}. \tag{3.5}$$

From (1.6), (3.4), and (3.5), we have

$$G(q)=q^{1/3}\frac{(q;q^2)_\infty}{(q^3;q^6)_\infty}=q^{1/3}\frac{\chi(-q)}{\chi^3(-q^3)}=q^{1/3}\frac{\psi(q^3)}{f(q,q^2)}. \tag{3.6}$$

Thus,

$$\frac{1}{\sqrt{G(q)}}+\sqrt{G(q)}=\frac{1}{q^{1/6}}.\sqrt{\frac{f(q,q^2)}{\psi(q^3)}}+q^{1/6}\sqrt{\frac{\psi(q^3)}{f(q,q^2)}}, \quad \text{using (3.6)}$$

$$=\frac{f(q,q^2)+q^{1/3}\psi(q^3)}{q^{1/6}\sqrt{\psi(q^3)f(q,q^2)}} \tag{3.7}$$

Using (3.3) and (3.4) in the numerator and the denominator of the right side of (3.7), we find that

$$\frac{1}{\sqrt{G(q)}}+\sqrt{G(q)}=\frac{\sqrt{\varphi(-q^3)}}{q^{1/6}\sqrt{\psi(q^3)}\varphi(-q^3)}\times\sqrt{\chi(-q)}\psi(q^{1/3}). \tag{3.8}$$

Employing (3.5) in the right side of (3.8), we arrive at (3.1).

Proof of (3.2). From [Berndt (1991),, p. 345, Entries 1(i), (ii)], we obtain

$$1-2G(q)=\frac{\varphi(-q^{1/3})}{q^{1/3}\varphi(-q^3)}. \tag{3.9}$$

Using (3.9)

$$\frac{1}{\sqrt{G(q)}}-2\sqrt{G(q)}=\frac{\varphi(-q^{1/3})}{q^{1/3}\varphi(-q^3)\sqrt{G(q)}}. \tag{3.10}$$

Employing (3.6) in the right side of (3.10), we arrive at (3.2).

Theorem 3.2. We have

$$1-G(q)+G^2(q)=G(q)\times\frac{\chi^3(-q^3)\chi(-q)\psi^4(q)}{q^{1/3}\phi^2(-q^3)\psi(q^3)\psi(q^{1/3})}. \tag{3.11}$$

$$1+2G(q)+4G^2(q)=G(q).\frac{\chi^3(-q^3)\phi^4(-q)}{q^{1/3}\chi(-q)\phi^3(-q^3)\phi(-q^{1/3})}. \tag{3.12}$$

Proof of (3.11). Here, we require (3.1). Note that for each i=1,2, we obtain an identity from (3.1) by replacing $q^{1/3}$ with $\omega^i q^{1/3}$, where $\omega = e^{2\pi i/3}$. Multiplying these two identities, we have

$$\prod_{i=1,2}\left(\frac{1}{\sqrt{\omega^i G(q)}}+\sqrt{\omega^i G(q)}\right)=\prod_{i=1,2}\frac{\sqrt{\chi^3(-q^3)\chi(-q)}}{\omega^{i/2}q^{1/6}\varphi(-q^3)}.\psi(\omega^i q^{1/3})=\frac{\chi^{3(}(-q^3)\chi(-q)}{q^{1/3}\varphi^2(-q^3)}.\psi(\omega q^{1/3})\psi(\omega^2 q^{1/3}). \tag{3.13}$$

Since

$$(1-q)(1-\omega q)(1-\omega^2 q)=1-q^3,$$

we have

$$(q;q)_\infty(\omega q;\omega q)_\infty(\omega^2 q;\omega^2 q)=\prod_{n=1}^{\infty}(1-q^n)(1-\omega^n q^n)(1-\omega^{2n}q^n)=\prod_{3/n}(1-q^n)^3\prod_{3\nmid n}(1-q^{3n})=\frac{(q^3;q^3)^2{}_\infty}{(q^9;q^9)_\infty}. \tag{3.14}$$

From (2.4)

$$\psi(q)=\frac{(q^2;q^2)^2{}_\infty}{(q;q)_\infty}. \tag{3.15}$$

Using (3.15) and (3.14), we find that

$$\begin{aligned}\psi(\omega q^{1/3})\psi(\omega^2 q^{1/3})&=\frac{(\omega^2 q^{2/3};\omega^2 q^{2/3})_\infty^2(\omega q^{2/3};\omega q^{2/3})_\infty^2}{(\omega q^{1/3};\omega q^{1/3})_\infty(\omega^2 q^{1/3};\omega^2 q^{1/3})_\infty}\\ &=\frac{(\omega^2 q^{2/3};\omega^2 q^{2/3})_\infty^2(\omega q^{2/3};\omega q^{2/3})_\infty^2}{(\omega q^{1/3};\omega q^{1/3})_\infty(\omega^2 q^{1/3};\omega^2 q^{1/3})_\infty}\times\frac{(q^{2/3};q^{2/3})_\infty^2(q^{1/3};q^{1/3})_\infty}{(q^{1/3};q^{1/3})_\infty(q^{2/3};q^{2/3})_\infty^2}\\ &=\frac{(q^2;q^2)_\infty^8(q^3;q^3)_\infty(q^{1/3};q^{1/3})_\infty}{(q^6;q^6)_\infty^2(q;q)_\infty^4(q^{2/3};q^{2/3})_\infty^2}\\ &=\frac{\psi^4(q)}{\psi(q^3)\psi(q^{1/3})}.\end{aligned} \tag{3.16}$$

Employing (3.16) in (3.13) and expanding the product on the left side, we obtain

$$\frac{1-G(q)+G^2(q)}{G(q)}=\frac{\chi^3(-q^3)\chi(-q)\psi^4(q)}{q^{1/3}\varphi^2(-q^3)\psi(q^3)\psi(q^{1/3})}. \tag{3.17}$$

Thus, we complete the proof of (3.11).

Proof of (3.12). Proof of the second part of the Theorem 3.2 is quite similar to the proof of first part. Here we require (3.2).

Proceeding as in the proof of first part, we deduce that

$$\prod_{i=1,2}\left(\frac{1}{\sqrt{\omega^i G(q)}}-2\sqrt{\omega^i G(q)}\right)=\frac{\chi^{3(}(-q^3)}{q^{1/3}\varphi^2(-q^3)\chi(-q)}.\varphi(-\omega q^{1/3})\varphi(-\omega^2 q^{1/3}). \tag{3.18}$$

Again, we have from (2.6)

$$\varphi(-q)=\frac{(q;q)^2{}_\infty}{(q^2;q^2)_\infty}. \tag{3.19}$$

Using (3.19) and (3.14), we obtain

$$\varphi(-\omega q^{1/3})\varphi(-\omega^2 q^{1/3}) = \frac{(\omega q^{1/3};\omega q^{1/3})_\infty^2(\omega^2 q^{1/3};\omega^2 q^{1/3})_\infty^2}{(\omega q^{2/3};\omega q^{2/3})_\infty(\omega q^{2/3};\omega q^{2/3})_\infty^2}$$
$$= \frac{(q;q)^8{}_\infty(q^6;q^6)_\infty(q^{2/3};q^{2/3})_\infty}{(q^2;q^2)_\infty^4(q^3;q^3)_\infty^2(q^{1/3};q^{1/3})_\infty^2}$$
$$= \frac{\varphi^4(-q)}{\varphi(-q^3)\varphi(-q^{1/3})}. \tag{3.20}$$

Now, expanding the left side of (3.18) and using (3.20), we obtain

$$\frac{1+2G(q)+4G^2(q)}{G(q)} = \cdot\frac{\chi^3(-q^3)\varphi^4(-q)}{q^{1/3}\chi(-q)\varphi^2(-q^3)\varphi(-q^3)\varphi(-q^{1/3})}, \tag{3.21}$$

which is equivalent to (3.12).

Corollary 3.3. We have

$$\frac{1}{\psi(q)\varphi(-q)} = \frac{q^2\varphi^5(-q^9)\psi(q^9)}{\chi^6(-q^9)\psi^4(q^3)\psi^4(-q^3)}\times\{4G^2(q^3)-2G(q^3)+3+1/G(q^3)+1/G^2(q^3)\}, \tag{3.22}$$

$$\frac{1}{\psi(q)\varphi^2(-q)} = \frac{q^3\chi(-q^3)\varphi^8(-q^9)\psi(q^9)}{\chi^9(-q^3)\psi^4(q^3)\psi^8(-q^3)}\times\begin{Bmatrix}16G^3(q^3)+12G(q^3)+8+9/G(q^3)\\ +3/G^2(q^3)+1/G^3(q^3)\end{Bmatrix}. \tag{3.23}$$

Proof. Multiplying (3.11) and (3.12) and replacing q by q3, we can easily derive (3.22). Squaring (3.12) and multiplying with (3.11) and replacing q by q3, we can easily derive (3.23).

Next, as corollary we obtain the following 3-dissections.

Corollary 3.4. We have the 3-dissection of $1/\psi(q)$ as

$$\frac{1}{\psi(q)} = \frac{\psi^3(q^9)}{\psi^4(q^3)\omega^2(q^3)} - q\frac{\psi^3(q^9)}{\psi^4(q^3)\omega(q^3)} + q^2\frac{\psi^3(q^9)}{\psi^4(q^3)}, \tag{3.25}$$

3-dissection of $1/\phi(q)$ as

$$\frac{1}{\phi(q)} = \frac{\varphi^3(-q^9)}{\varphi^4(-q^3)} + 2q\frac{\varphi^3(-q^9)\omega(q^3)}{\varphi^4(-q^3)} + 4q^2\frac{\varphi^3(-q^9)\omega^2(q^3)}{\varphi^4(-q^3)}, \tag{3.26}$$

3-dissection of $1/(q;q)_\infty(q^2;q^2)_\infty$ as

$$\frac{1}{(q;q)_\infty(q^2;q^2)_\infty} = \frac{(q^9;q^9)^3{}_\infty(q^{18};q^{18})^3{}_\infty}{(q^3;q^3)^4{}_\infty(q^6;q^6)^4{}_\infty}\times\left\{\frac{1}{\omega^2(q^3)}+\frac{q}{\omega^2(q^3)}+3q^2-2q^3\omega(q^3)+4q^4\omega^3(q^3)\right\}, \tag{3.26}$$

3-dissection of $1/(q;q)^3{}_\infty$ as

$$\frac{1}{(q;q)_\infty^3} = \frac{(q^9;q^9)_\infty^9}{(q^3;q^3)_\infty^{12}}\times\left\{\frac{1}{\omega^2(q^3)}+3\frac{q}{\omega^2(q^3)}+9q^2+8q^3\omega(q^3)+12q^4\omega^3(q^3)+16q^6\omega^4(q^3)\right\}, \tag{3.27}$$

where

$$G(q) = q^{1/3}\frac{(q;q^2)_\infty}{(q^3;q^6)_\infty^3} = q^{1/3}\omega(q). \tag{3.28}$$

Proof. Employing (3.28) in (3.11) and replacing q by q3, we arrive at (3.24). Employing (3.28) in (3.12) and replacing q by q3, we find (3.25). In (3.22), employing (3.28) in the right hand side andusing (3.15), (3.19) in the left hand side, we obtain (3.26). In (3.23), employing (3.28) in the right side and using (3.15), (3.19) in the left side, we arrive at (3.27).

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