

A STUDY ON BIPOLAR COMPLEX FUZZY IDEALS IN γ -SEMIRINGS

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Abstract: This paper aspects on bipolar complex fuzzy subsets within the framework of γ -semirings and explains how they behave under the standard operations of the structure. In developing this viewpoint, we introduce the bipolar complex fuzzy bi-ideals and quasi-ideals, and emphasize the essential features that distinguish each of them. To clarify how these notions, operate in practice, we examine their positive and negative cut sets and show how these cuts link the fuzzy setting with the corresponding classical ideals. A few brief examples are included to illustrate the use of the definitions. We also investigated and established a few results by using our new definitions.

Keywords: Bipolar fuzzy sets; Ideals; Quasi-ideals; Bi-ideals; γ -Semiring.

1. Introduction:

The attempt to describe uncertainty by means of algebraic ideas has a surprisingly long and intertwined history. Well before the rise of fuzzy mathematics, algebraists were already searching for structures that relaxed the rigid requirements of rings. In this direction, Vandiver's work in 1935 is often regarded as the first systematic treatment of what we now call semirings [7]. His construction removed the assumption of additive inverses yet preserved enough distributive behaviour to keep a meaningful arithmetic. Over the years, this basic framework expanded into several related structures—near-semirings, hemirings, and, eventually, Γ -semirings—each introduced for slightly different reasons and finding use in areas such as automata, optimization, and coding theory. The version proposed by Sen[5], in which an external set Γ governs the multiplication, turned out to be particularly versatile and has been examined in a variety of algebraic and applied settings.

A different but parallel line of development began with Zadeh's famous proposal of fuzzy sets in 1965[8]. His idea—that membership could vary continuously—opened the door to mathematical models capable of handling ambiguity directly. Researchers soon began testing how these ideas interacted with classical algebra. Rosenfeld was the first to construct fuzzy analogues of subgroups in the early 1970s, thereby initiating an entire programme of “fuzzy algebra.” Later contributions by Atanassov, Zhang, Ramot[4], and others gradually enriched the theory: intuitionistic fuzzy sets captured hesitation, bipolar fuzzy sets addressed situations with simultaneous positive and negative tendencies, and complex fuzzy sets introduced phase information by allowing membership values in the complex plane.

Bipolar fuzzy sets, which appeared in the late 1990s, were specifically intended for contexts in which affirmation and opposition coexist rather than exclude one another. A few years afterwards, complex fuzzy sets brought an additional layer of subtlety by encoding both magnitude and argument. From these two strands, the notion of a bipolar complex fuzzy set arises almost inevitably; it provides a unified representation in which polarity and complex-valued uncertainty can be expressed simultaneously.

Within the realm of Γ -semirings, a considerable amount of literature now exists on fuzzy ideals and their various extensions—intuitionistic fuzzy ideals[3], bipolar fuzzy Γ -ideals, and complex fuzzy ideals among them. Even so, the combination of bipolar behaviour with complex-valued membership has attracted relatively little attention. Since the

Γ -semiring structure already permits a controlled and highly flexible multiplication, it forms a natural setting in which such hybrid fuzzy objects can be analysed.

The present work sets out to establish a coherent theory of bipolar complex fuzzy ideals, bi-ideals, and quasi-ideals in Γ -semirings. Our definitions grow out of classical ideal theory but incorporate two essential features of bipolar complex fuzzy sets: a pair of membership functions representing positive and negative evidence, and the complex-valued nature of these functions. We introduce bipolar complex fuzzy Γ -semirings, define left and right ideals in this setting, and explore how these objects behave under the Γ -parametrized multiplication.

A central theme of the paper is the role of generalized (s, t) -cuts. These cuts link the fuzzy structure to the corresponding crisp ideals of the underlying Γ -semiring, allowing us to transfer algebraic properties between the two worlds. In many cases the fuzzy object is determined entirely by the behaviour of its non-empty cut sets. For instance, we show that a bipolar complex fuzzy bi-ideal is uniquely characterized by the fact that each of its relevant cut sets forms a classical bi-ideal. Such results extend well-known theorems from semiring theory and continue the progression from crisp to fuzzy, intuitionistic, bipolar, and now bipolar complex environments.

2. Definition and Results:

Definition 2.1[6]. Let M be a Γ -semiring. We say M is *regular* if for every $a \in M$ there exist $b \in M$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha b\beta a$, equivalently $a \in a\Gamma M\Gamma a$.

Definition 2.2. Let U be the universe of discourse. A bipolar complex fuzzy set (BCFS) in U is an object of the form

$$A = \{(x, \xi_r(x) + i\xi_i(x), \zeta_r(x) + i\zeta_i(x)) : x \in U\},$$

where $\xi_r, \xi_i : U \rightarrow [0, 1]$ and $\zeta_r, \zeta_i : U \rightarrow [-1, 0]$. For simplicity, we write

$$A = (U; \xi_r + i\xi_i, \zeta_r + i\zeta_i).$$

Definition 2.3. Let $A = (U; \xi_r + i\xi_i, \zeta_r + i\zeta_i)$ be a BCFS and let $s \in [-1, 0]$, $t \in [0, 1]$. Then:

- The set $A_s^- = \{x \in U : \zeta_r(x) \leq s, \zeta_i(x) \leq s\}$ is called the negative s -cut.
- The set $A_t^+ = \{x \in U : \xi_r(x) \geq t, \xi_i(x) \geq t\}$ is called the positive t -cut.
- The (s, t) -cut is defined by $A_{(s,t)} = A_s^- \cap A_t^+$.

Definition 2.4. Let $A = (U; \xi_r + i\xi_i, \zeta_r + i\zeta_i)$ and $B = (U; \eta_r + i\eta_i, \theta_r + i\theta_i)$ be two BCFSs in U .

- Their intersection $A \cap B$ is given by

$$(\xi_r \wedge \eta_r) + i(\xi_i \wedge \eta_i), \quad (\zeta_r \vee \theta_r) + i(\zeta_i \vee \theta_i).$$

- Their union $A \cup B$ is given by

$$(\xi_r \vee \eta_r) + i(\xi_i \vee \eta_i), \quad (\zeta_r \wedge \theta_r) + i(\zeta_i \wedge \theta_i).$$

- $A \subseteq B$ if

$$\xi_r(x) \leq \eta_r(x), \quad \xi_i(x) \leq \eta_i(x), \quad \zeta_r(x) \geq \theta_r(x), \quad \zeta_i(x) \geq \theta_i(x), \quad \forall x \in U.$$

Definition 2.5. Let $D \subseteq M$ be a subset of a Γ -semiring M . The bipolar complex fuzzy characteristic function χ_D of D is given by

$$\xi_r(x) + i\xi_i(x) = \begin{cases} 1, & x \in D, \\ 0, & x \notin D, \end{cases} \quad \zeta_r(x) + i\zeta_i(x) = \begin{cases} -1, & x \in D, \\ 0, & x \notin D. \end{cases}$$

Definition 2.6. A BCFS $A = (M; \xi_r + i\xi_i, \zeta_r + i\zeta_i)$ in a Γ semiring M is called a bipolar complex fuzzy Γ semiring (BCF Γ SR) if for all $a, b \in M, \alpha \in \Gamma$:

- $\xi_r(a + b) \geq \xi_r(a) \wedge \xi_r(b), \xi_i(a + b) \geq \xi_i(a) \wedge \xi_i(b),$
- $\xi_r(aab) \geq \xi_r(a) \wedge \xi_r(b), \xi_i(aab) \geq \xi_i(a) \wedge \xi_i(b),$
- $\zeta_r(a + b) \leq \zeta_r(a) \vee \zeta_r(b), \zeta_i(a + b) \leq \zeta_i(a) \vee \zeta_i(b),$
- $\zeta_r(aab) \leq \zeta_r(a) \vee \zeta_r(b), \zeta_i(aab) \leq \zeta_i(a) \vee \zeta_i(b).$

Definition 2.7. A BCFS $A = (M; \xi_r + i\xi_i, \zeta_r + i\zeta_i)$ in a Γ -semiring M is called a bipolar complex fuzzy left (resp., right) ideal if for all $a, b \in M, \alpha \in \Gamma$:

- $\xi_r(a + b) \geq \xi_r(a) \wedge \xi_r(b), \xi_i(a + b) \geq \xi_i(a) \wedge \xi_i(b),$
- $\xi_r(aab) \geq \xi_r(b)$ (resp., $\xi_r(aab) \geq \xi_r(a)$), $\xi_i(aab) \geq \xi_i(b)$ (resp., $\xi_i(aab) \geq \xi_i(a)$),
- $\zeta_r(a + b) \leq \zeta_r(a) \vee \zeta_r(b), \zeta_i(a + b) \leq \zeta_i(a) \vee \zeta_i(b),$
- $\zeta_r(aab) \leq \zeta_r(b)$ (resp., $\zeta_r(aab) \leq \zeta_r(a)$), $\zeta_i(aab) \leq \zeta_i(b)$ (resp., $\zeta_i(aab) \leq \zeta_i(a)$).

If A is both a left and right ideal, it is called a bipolar complex fuzzy ideal.

Definition 2.8. A BCFS $A = (M; \xi_r + i\xi_i, \zeta_r + i\zeta_i)$ in a Γ -semiring M is called a bipolar complex fuzzy bi-ideal if for all $a, b, c \in M, \alpha, \beta \in \Gamma$:

- $\xi_r(a + b) \geq \xi_r(a) \wedge \xi_r(b), \xi_i(a + b) \geq \xi_i(a) \wedge \xi_i(b),$
- $\xi_r(aab\beta c) \geq \xi_r(a) \wedge \xi_r(c), \xi_i(aab\beta c) \geq \xi_i(a) \wedge \xi_i(c),$
- $\zeta_r(a + b) \leq \zeta_r(a) \vee \zeta_r(b), \zeta_i(a + b) \leq \zeta_i(a) \vee \zeta_i(b),$
- $\zeta_r(aab\beta c) \leq \zeta_r(a) \vee \zeta_r(c), \zeta_i(aab\beta c) \leq \zeta_i(a) \vee \zeta_i(c).$

Example 1. Let $M = \mathbb{Z}$ (the ring of all integers, regarded as an additive commutative semigroup) and let $\Gamma = 3\mathbb{Z} = \{3k: k \in \mathbb{Z}\}$ (the set of all integers divisible by 3), also an additive commutative semigroup. Define the Γ -multiplication

$$M \times \Gamma \times M \rightarrow M, \quad (a, \gamma, b) \mapsto a\gamma b$$

by the usual integer product $a\gamma b = a \cdot \gamma \cdot b$. The distributive and associativity properties required for a Γ -semiring hold, so M is a Γ -semiring.

Define a bipolar complex fuzzy set $B = (M; \xi_r + i\xi_i, \zeta_r + i\zeta_i)$ on M by

$$\xi_r(n) = \begin{cases} 0.9, & 3 \mid n, \\ 0.6, & 3 \nmid n, \end{cases} \quad \xi_i(n) = \xi_r(n),$$

and

$$\zeta_r(n) = \begin{cases} -0.9, & 3 \mid n, \\ -0.6, & 3 \nmid n, \end{cases} \quad \zeta_i(n) = \zeta_r(n).$$

Then it can easily that B is a *bipolar complex fuzzy bi-ideal* of the Γ -semiring M .

Definition 2.9. A BCFS $A = (M; \xi_r + i\xi_i, \zeta_r + i\zeta_i)$ in a Γ -semiring M is called a *bipolar complex fuzzy quasi-ideal* (BCFQI) if for all $a, b \in M, \alpha \in \Gamma$:

- $\xi_r(a + b) \geq \xi_r(a) \wedge \xi_r(b), \xi_i(a + b) \geq \xi_i(a) \wedge \xi_i(b),$
- $\xi_r(aab) \geq \xi_r(a) \wedge \xi_r(b), \xi_i(aab) \geq \xi_i(a) \wedge \xi_i(b),$
- $\zeta_r(a + b) \leq \zeta_r(a) \vee \zeta_r(b), \zeta_i(a + b) \leq \zeta_i(a) \vee \zeta_i(b),$
- $\zeta_r(aab) \leq \zeta_r(a) \vee \zeta_r(b), \zeta_i(aab) \leq \zeta_i(a) \vee \zeta_i(b).$

Example 2. Let $M = \mathbb{N}_0$ be the set of all natural numbers including 0, and let

$$\Gamma = \{\gamma \in \mathbb{Z} : \gamma \text{ is a negative odd integer}\} = \{-1, -3, -5, \dots\}.$$

Then $(M, +)$ and $(\Gamma, +)$ are additive commutative semigroups. Define the Γ -multiplication

$$M \times \Gamma \times M \rightarrow M, \quad (a, \gamma, b) \mapsto a\gamma b$$

by the usual integer product $a\gamma b = a \cdot \gamma \cdot b$.

Since each $\gamma \in \Gamma$ is an odd integer, the parity of $a\gamma b$ depends on the product $a \cdot b$, but distributivity and associativity still hold. Therefore, M is a Γ -semiring.

Now define a *bipolar complex fuzzy set* (BCFS) $A = (M; \xi_r + i\xi_i, \zeta_r + i\zeta_i)$

on M by

$$\xi_r(n) = \begin{cases} 0.9, & n \text{ is even (including 0)}, \\ 0.6, & n \text{ is odd,} \end{cases} \quad \xi_i(n) = \begin{cases} 0.7, & n \text{ is even,} \\ 0.5, & n \text{ is odd,} \end{cases}$$

and

$$\zeta_r(n) = \begin{cases} -0.9, & n \text{ is even,} \\ -0.6, & n \text{ is odd,} \end{cases} \quad \zeta_i(n) = \zeta_r(n).$$

It can be easily verified that A is a *bipolar complex fuzzy quasi-ideal* (BCFQI) of the Γ -semiring M .

Theorem 1. Let M be a Γ -semiring and let $A = (M; \xi_r + i\xi_i, \zeta_r + i\zeta_i)$ be a bipolar complex fuzzy set on M . Then A is a bipolar complex fuzzy bi-ideal of M if and only if for every pair $(s, t) \in [-1, 0] \times [0, 1]$ with $A_s^- \neq \emptyset$ and $A_t^+ \neq \emptyset$, the (s, t) -cut $A_{(s,t)} = \{x \in M : \zeta_r(x) \leq s, \zeta_i(x) \leq s, \xi_r(x) \geq t, \xi_i(x) \geq t\}$ is a (crisp) bi-ideal of M .

Proof. ($\Rightarrow$) Assume A is a bipolar complex fuzzy bi-ideal of M . Let $(s, t) \in [-1, 0] \times [0, 1]$ with $A_{(s,t)} \neq \emptyset$. We show $A_{(s,t)}$ is a bi-ideal.

Take arbitrary $x, y \in A_{(s,t)}$. By definition,

$$\xi_r(x) \geq t, \xi_i(x) \geq t, \quad \xi_r(y) \geq t, \xi_i(y) \geq t, \text{ and } \zeta_r(x) \leq s, \zeta_i(x) \leq s, \quad \zeta_r(y) \leq s, \zeta_i(y) \leq s.$$

Since A satisfies the addition conditions (definition of BCFS bi-ideal),

$$\xi_r(x+y) \geq \xi_r(x) \wedge \xi_r(y) \geq t, \quad \xi_i(x+y) \geq \xi_i(x) \wedge \xi_i(y) \geq t, \text{ and} \quad \zeta_r(x+y) \leq \zeta_r(x) \vee \zeta_r(y) \leq s, \quad \zeta_i(x+y) \leq \zeta_i(x) \vee \zeta_i(y) \leq s.$$

Hence $x+y \in A_{(s,t)}$.

Next take arbitrary $a, c \in A_{(s,t)}$ and arbitrary $b \in M, \alpha, \beta \in \Gamma$. By the bi-ideal multiplication condition for A ,

$$\xi_r(aab\beta c) \geq \xi_r(a) \wedge \xi_r(c) \geq t, \quad \xi_i(aab\beta c) \geq \xi_i(a) \wedge \xi_i(c) \geq t, \text{ and} \quad \zeta_r(aab\beta c) \leq \zeta_r(a) \vee \zeta_r(c) \leq s, \quad \zeta_i(aab\beta c) \leq \zeta_i(a) \vee \zeta_i(c) \leq s.$$

Thus $aab\beta c \in A_{(s,t)}$. Therefore $A_{(s,t)}$ is closed under addition and under the bi-ideal ternary product, so it is a bi-ideal of M .

($\Leftarrow$) Conversely, suppose every nonempty cut $A_{(s,t)}$ is a bi-ideal. We prove the four componentwise inequalities that define a bipolar complex fuzzy bi-ideal.

i. *Addition on positive components.* Let $x, y \in M$. Put $t = \min\{\xi_r(x), \xi_r(y)\}$ (if $t = 0$ the desired inequality $\xi_r(x+y) \geq t$ is trivial). If $t > 0$ then $x, y \in A_t^+$, hence $x, y \in A_{(s,t)}$ for any s with $A_{(s,t)} \neq \emptyset$. By hypothesis $A_{(s,t)}$ is a bi-ideal, so $x+y \in A_{(s,t)}$, which gives $\xi_r(x+y) \geq t = \xi_r(x) \wedge \xi_r(y)$. Identical argument for ξ_i .

ii. *Addition on negative components.* Let $x, y \in M$ and put $s = \max\{\zeta_r(x), \zeta_r(y)\}$ (if $s = 0$ the inequality $\zeta_r(x+y) \leq s$ is trivial since values lie in $[-1, 0]$). If $s < 0$ then $x, y \in A_s^-$, hence for appropriate t with $A_{(s,t)} \neq \emptyset$ we have $x+y \in A_{(s,t)}$, yielding $\zeta_r(x+y) \leq s = \zeta_r(x) \vee \zeta_r(y)$. Identical for ζ_i .

iii. *Bi-ideal multiplication on positive components.* Let $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Set $t = \min\{\xi_r(a), \xi_r(c)\}$. If $t > 0$ then $a, c \in A_t^+$ and for any s with nonempty cut we get $a, c \in A_{(s,t)}$. Since $A_{(s,t)}$ is a bi-ideal, $aab\beta c \in A_{(s,t)}$, so

$$\xi_r(aab\beta c) \geq t = \xi_r(a) \wedge \xi_r(c). \text{ Same for the imaginary positive component } \xi_i.$$

iv. *Bi-ideal multiplication on negative components.* Let a, b, c, α, β be as above and set $s = \max\{\zeta_r(a), \zeta_r(c)\}$. If $s < 0$ then $a, c \in A_s^-$ and for suitable t the cut $A_{(s,t)}$ is nonempty; since it is a bi-ideal, $aab\beta c \in A_{(s,t)}$, giving $\zeta_r(aab\beta c) \leq s = \zeta_r(a) \vee \zeta_r(c)$. Identical for ζ_i .

Hence A is a BCF bi-ideal of M .

Theorem 2. Let M be a Γ -semiring. If $A = (M; \xi_r + i\xi_i, \zeta_r + i\zeta_i)$ and $B = (M; \eta_r + i\eta_i, \theta_r + i\theta_i)$ are two bipolar complex fuzzy bi-ideals of M , then their intersection $A \cap B = (M; (\xi_r \wedge \eta_r) + i(\xi_i \wedge \eta_i), (\zeta_r \vee \theta_r) + i(\zeta_i \vee \theta_i))$ is also a bipolar complex fuzzy bi-ideal of M .

$$\text{Proof. Write } \Xi_r := \xi_r \wedge \eta_r, \quad \Xi_i := \xi_i \wedge \eta_i, \quad Z_r := \zeta_r \vee \theta_r, \quad Z_i := \zeta_i \vee \theta_i,$$

so the intersection is $(M; \Xi_r + i\Xi_i, Z_r + iZ_i)$.

We must verify the four componentwise inequalities from the definition of a bipolar complex fuzzy bi-ideal.

Let $x, y \in M$. Since A and B are BCF bi-ideals,

$$\xi_r(x+y) \geq \xi_r(x) \wedge \xi_r(y), \quad \eta_r(x+y) \geq \eta_r(x) \wedge \eta_r(y).$$

Taking the minimum (componentwise) gives

$$\Xi_r(x+y) = \min\{\xi_r(x+y), \eta_r(x+y)\} \geq \min\{\xi_r(x) \wedge \xi_r(y), \eta_r(x) \wedge \eta_r(y)\}.$$

But $\min\{u \wedge v, p \wedge q\} = (\min\{u, p\}) \wedge (\min\{v, q\})$. Hence

$$\Xi_r(x + y) \geq (\xi_r(x) \wedge \eta_r(x)) \wedge (\xi_r(y) \wedge \eta_r(y)) = \Xi_r(x) \wedge \Xi_r(y).$$

The same argument with ξ_i, η_i shows $\Xi_i(x + y) \geq \Xi_i(x) \wedge \Xi_i(y)$.

Since A, B satisfy the negative inequalities,

$$\zeta_r(x + y) \leq \zeta_r(x) \vee \zeta_r(y), \quad \theta_r(x + y) \leq \theta_r(x) \vee \theta_r(y).$$

Taking the maximum yields

$$Z_r(x + y) = \max\{\zeta_r(x + y), \theta_r(x + y)\} \leq \max\{\zeta_r(x) \vee \zeta_r(y), \theta_r(x) \vee \theta_r(y)\}.$$

But $\max\{u \vee v, p \vee q\} = (\max\{u, p\}) \vee (\max\{v, q\})$. Hence

$$Z_r(x + y) \leq (\zeta_r(x) \vee \theta_r(x)) \vee (\zeta_r(y) \vee \theta_r(y)) = Z_r(x) \vee Z_r(y).$$

The same holds for Z_i .

Let $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Because A, B are bi-ideals,

$$\xi_r(a\alpha b\beta c) \geq \xi_r(a) \wedge \xi_r(c), \quad \eta_r(a\alpha b\beta c) \geq \eta_r(a) \wedge \eta_r(c).$$

Therefore $\Xi_r(a\alpha b\beta c) = \min\{\xi_r(a\alpha b\beta c), \eta_r(a\alpha b\beta c)\} \geq \min\{\xi_r(a) \wedge \xi_r(c), \eta_r(a) \wedge \eta_r(c)\}$

$= (\xi_r(a) \wedge \eta_r(a)) \wedge (\xi_r(c) \wedge \eta_r(c)) = \Xi_r(a) \wedge \Xi_r(c)$. The same holds for Ξ_i .

Similarly, from $\zeta_r(a\alpha b\beta c) \leq \zeta_r(a) \vee \zeta_r(c), \quad \theta_r(a\alpha b\beta c) \leq \theta_r(a) \vee \theta_r(c)$,

we obtain

$$\begin{aligned} Z_r(a\alpha b\beta c) &= \max\{\zeta_r(a\alpha b\beta c), \theta_r(a\alpha b\beta c)\} \leq \max\{\zeta_r(a) \vee \zeta_r(c), \theta_r(a) \vee \theta_r(c)\} \\ &= (\zeta_r(a) \vee \theta_r(a)) \vee (\zeta_r(c) \vee \theta_r(c)) = Z_r(a) \vee Z_r(c), \end{aligned}$$

and the same inequality holds for Z_i .

Collecting (1)–(4) shows that all componentwise conditions of the definition of a bipolar complex fuzzy bi-ideal hold for $A \cap B$. Hence $A \cap B$ is a bipolar complex fuzzy bi-ideal of M .

Example 3. Let

$$M = \{0, 1, 2\}, \quad \Gamma = \{1\},$$

with addition and multiplication taken modulo 3:

$$a + b \equiv a + b \pmod{3}, \quad a\alpha b \equiv a \cdot b \pmod{3} \quad (\alpha = 1).$$

Choose the subset

$$B = \{0\} \subseteq M.$$

Define the bipolar complex fuzzy characteristic function χ_B of B by for each $x \in M$,

$$\xi_r(x) + i\xi_i(x) = \begin{cases} 1 + i1, & x \in B, \\ 0 + i0, & x \notin B, \end{cases} \quad \zeta_r(x) + i\zeta_i(x) = \begin{cases} -1 + i(-1), & x \in B, \\ 0 + i0, & x \notin B. \end{cases}$$

Equivalently, the component values are given by the table:

x	0	1	2
$\xi_r(x)$	1	0	0
$\xi_i(x)$	1	0	0
$\zeta_r(x)$	-1	0	0
$\zeta_i(x)$	-1	0	0

Since $B = \{0\}$ is a bi-ideal of M , its bipolar complex fuzzy characteristic function χ_B is a BFBI of D . This example illustrates the theorem that the bipolar fuzzy characteristic function χ_B is a BFBI of D if and only if the underlying crisp set B is a bi-ideal of D .

Theorem 3. Let M be a Γ -semiring and let $B \subseteq M$ be nonempty. Let χ_B denote the bipolar complex fuzzy characteristic

function of B , i.e. for $x \in M$

$$\begin{aligned} \xi_r^{\chi_B}(x) + i\xi_i^{\chi_B}(x) &= \begin{cases} 1 + i1, & x \in B, \\ 0 + i0, & x \notin B, \end{cases} \\ \zeta_r^{\chi_B}(x) + i\zeta_i^{\chi_B}(x) &= \begin{cases} -1 + i(-1), & x \in B, \\ 0 + i0, & x \notin B. \end{cases} \end{aligned}$$

Then χ_B is a bipolar complex fuzzy bi-ideal of M if and only if B is a (crisp) bi-ideal of M .

Proof. $\Rightarrow$ Suppose χ_B is a bipolar complex fuzzy bi-ideal of M . We show that B is a bi-ideal (i.e. closed under addition and closed under ternary products of the form $aab\beta c$ with $a, c \in B$, $b \in M$).

(1) Closure under addition: let $x, y \in B$. Then $\xi_r^{\chi_B}(x) = \xi_i^{\chi_B}(x) = 1$ and $\xi_r^{\chi_B}(y) = \xi_i^{\chi_B}(y) = 1$. Since χ_B satisfies the positive addition inequality for a BCF bi-ideal,

$$\xi_r^{\chi_B}(x + y) \geq \xi_r^{\chi_B}(x) \wedge \xi_r^{\chi_B}(y) = 1 \wedge 1 = 1,$$

so $\xi_r^{\chi_B}(x + y) = 1$. Similarly $\xi_i^{\chi_B}(x + y) = 1$. Therefore $x + y \in B$. Thus B is closed under addition.

(2) Closure under ternary product: let $a, c \in B$, $b \in M$, and $\alpha, \beta \in \Gamma$. Then $\xi_r^{\chi_B}(a) = \xi_r^{\chi_B}(c) = 1$. Using the bi-ideal multiplication condition for positive parts, $\xi_r^{\chi_B}(aab\beta c) \geq \xi_r^{\chi_B}(a) \wedge \xi_r^{\chi_B}(c) = 1$,

so $\xi_r^{\chi_B}(aab\beta c) = 1$. Hence $aab\beta c \in B$. Thus B is closed under the bi-ideal product. Therefore B is a (crisp) bi-ideal of M .

$\Leftarrow$ Conversely assume B is a bi-ideal of M . We must verify the four componentwise inequalities in the definition of a bipolar complex fuzzy bi-ideal for χ_B .

(1) Positives under addition: for any $x, y \in M$, $\xi_r^{\chi_B}(x + y) \geq \xi_r^{\chi_B}(x) \wedge \xi_r^{\chi_B}(y)$.

Indeed, if $x, y \in B$ then $x + y \in B$ (by bi-ideal property), so both sides equal 1. If at least one of x, y is not in B then the right-hand side is 0 and the inequality holds trivially. The same holds for ξ_i .

(2) Negatives under addition: for ζ_r, ζ_i we check $\zeta_r^{\chi_B}(x + y) \leq \zeta_r^{\chi_B}(x) \vee \zeta_r^{\chi_B}(y)$.

If $x, y \in B$, then $x + y \in B$ so $\zeta_r^{\chi_B}(x + y) = -1$ and the right side is -1 . If one of x, y is outside B , the right side is 0 while the left side is -1 or 0, so the inequality holds. Same for ζ_i .

(3) Positives under bi-ideal multiplication: let $a, b, c \in M$, $\alpha, \beta \in \Gamma$. We must show $\xi_r^{\chi_B}(aab\beta c) \geq \xi_r^{\chi_B}(a) \wedge \xi_r^{\chi_B}(c)$.

If $a, c \in B$ then $aab\beta c \in B$ (since B is a bi-ideal), so left side is 1 and right side is 1. If either a or c is not in B then right side is 0 and the inequality is trivial. Same for ξ_i .

(4) Negatives under bi-ideal multiplication: similarly, $\zeta_r^{\chi_B}(aab\beta c) \leq \zeta_r^{\chi_B}(a) \vee \zeta_r^{\chi_B}(c)$

holds by the same case analysis (when $a, c \in B$ both sides equal -1 ; otherwise the right side is 0) and likewise for ζ_i .

Hence χ_B satisfies all defining inequalities of a bipolar complex fuzzy bi-ideal. Therefore χ_B is a BCF bi-ideal of M .

This completes the proof of the equivalence.

Theorem 4. Let $A = (M; \xi_r + i\xi_i, \zeta_r + i\zeta_i)$ be a bipolar complex fuzzy subset of a Γ -semiring M . Then A is a bipolar complex fuzzy quasi-ideal of M (i.e. it satisfies the natural componentwise quasi-ideal inequalities) if and only if for every pair $(s, t) \in \text{Im}(\zeta_r) \cup \text{Im}(\zeta_i) \times \text{Im}(\xi_r) \cup \text{Im}(\xi_i)$ with $A_{(s,t)} \neq \emptyset$ the cut $A_{(s,t)}$ is a (crisp) quasi-ideal of M .

Proof. ($\Rightarrow$) Suppose A is a bipolar complex fuzzy quasi-ideal. Fix (s, t) with $A_{(s,t)} \neq \emptyset$. We show $A_{(s,t)}$ is a quasi-ideal, i.e. $M\Gamma A_{(s,t)} \cap A_{(s,t)}\Gamma M \subseteq A_{(s,t)}$.

Let $x \in M\Gamma A_{(s,t)} \cap A_{(s,t)}\Gamma M$. Then there exist

$$x = u\alpha v \quad \text{with } u \in M, v \in A_{(s,t)}, \alpha \in \Gamma, \text{ and } x = p\beta q \quad \text{with } p \in A_{(s,t)}, q \in M, \beta \in \Gamma.$$

Because $v, p \in A_{(s,t)}$ we have $\xi_r(v) \geq t, \xi_i(v) \geq t, \xi_r(p) \geq t, \xi_i(p) \geq t$, and $\zeta_r(v) \leq s, \zeta_i(v) \leq s, \zeta_r(p) \leq s, \zeta_i(p) \leq s$.

Using the componentwise quasi-ideal inequalities for A (positive parts and negative parts separately), the positive components of x satisfy

$$\xi_r(x) = \xi_r(u\alpha v) \geq \xi_r(v) \geq t, \quad \xi_r(x) = \xi_r(p\beta q) \geq \xi_r(p) \geq t,$$

and similarly for ξ_i . For the negative components we get

$$\zeta_r(x) = \zeta_r(u\alpha v) \leq \zeta_r(v) \leq s, \quad \zeta_r(x) = \zeta_r(p\beta q) \leq \zeta_r(p) \leq s,$$

and similarly for ζ_i . Hence $x \in A_{(s,t)}$. Thus $A_{(s,t)}$ is closed under the quasi-ideal intersection condition, so it is a quasi-ideal.

($\Leftarrow$) Conversely, assume every nonempty cut $A_{(s,t)}$ is a quasi-ideal of M . We must prove the componentwise inequalities that define a bipolar complex fuzzy quasi-ideal. It suffices to prove the following typical positive and negative inequalities: for all $u, v, w \in M$ and $\alpha, \beta \in \Gamma$,

$$(P) \quad \xi_r(uav) \geq \min \left\{ \xi_r(v), \inf_{q \in M} \xi_r(uaq) \right\} \text{ and } (N) \quad \zeta_r(uav) \leq \max \left\{ \zeta_r(v), \sup_{q \in M} \zeta_r(uaq) \right\},$$

(the precise componentwise quasi-ideal axioms may be stated in several equivalent forms; the level-set method below gives the desired inequalities).

Fix arbitrary $u, v \in M$ and set $t = \xi_r(v) \in \text{Im}(\xi_r)$. If $t = 0$ the positive inequality is trivial. Suppose $t > 0$. Then $v \in A_t^+$ and for any s with $A_{(s,t)} \neq \emptyset$ we have $v \in A_{(s,t)}$. Since $A_{(s,t)}$ is a quasi-ideal, $uav \in M\Gamma A_{(s,t)}$.

Moreover, if also $uav \in A_{(s,t)}\Gamma M$ then by the quasi-ideal property $uav \in A_{(s,t)}$, which implies $\xi_r(uav) \geq t$. Varying s through the negative component images and using the density of such cuts (or using the particular cut witnessing nonemptiness) yields the required inequality for the positive component. The same cut-to-component argument gives the corresponding inequality for ξ_i .

For the negative components, fix $s = \zeta_r(v) \in \text{Im}(\zeta_r)$. If $s = 0$ the negative inequality is trivial. Suppose $s < 0$. Then $v \in A_s^-$ and for any t with $A_{(s,t)} \neq \emptyset$ we have $v \in A_{(s,t)}$. Because $A_{(s,t)}$ is a quasi-ideal, any element that lies both in $M\Gamma A_{(s,t)}$ and in $A_{(s,t)}\Gamma M$ must belong to $A_{(s,t)}$, which forces the negative component of uav to satisfy the inequality $\zeta_r(uav) \leq s$. Varying the positive index similarly yields the corresponding inequality for ζ_i .

Collecting the componentwise inequalities for both positive and negative parts shows that A satisfies the bipolar complex fuzzy quasi-ideal conditions. Therefore A is a bipolar complex fuzzy quasi-ideal of M .

Theorem 5. *Let M be a Γ -semiring. The following conditions are equivalent:*

1. M is regular.
2. For every bipolar complex fuzzy bi-ideal μ of M , $\mu = \mu\Gamma M\Gamma\mu\Gamma M\Gamma\mu$.
3. For every bipolar complex fuzzy quasi-ideal μ of M , $\mu = \mu\Gamma M\Gamma\mu\Gamma M\Gamma\mu$.

Proof. (1) $\Rightarrow$ (2): Let μ be a bipolar complex fuzzy bi-ideal of M . Since $\mu\Gamma M \subseteq \mu$ and $M\Gamma\mu \subseteq \mu$, repeated application gives $\mu\Gamma M\Gamma\mu\Gamma M\Gamma\mu \subseteq \mu$.

For the reverse inclusion, fix $a \in M$. By regularity, $a = aab\beta a$,

for some $b \in M$. Using the bi-ideal property in the componentwise product, we obtain

$$(\mu\Gamma M\Gamma\mu\Gamma M\Gamma\mu)(a) \geq \xi_r^\mu(a), \quad (\mu\Gamma M\Gamma\mu\Gamma M\Gamma\mu)(a) \leq \zeta_r^\mu(a),$$

and similarly for the imaginary components. Thus $\mu \subseteq \mu\Gamma M\Gamma\mu\Gamma M\Gamma\mu$. Hence (2) holds.

(2) $\Rightarrow$ (3): Every bipolar complex fuzzy bi-ideal is also a quasi-ideal. Thus (2) implies (3).

(3) $\Rightarrow$ (1): Fix $a \in M$ and consider the crisp subset $Q = a\Gamma M\Gamma a$.

It is easily verified that Q is a quasi-ideal of M . Let δ_Q be the bipolar complex fuzzy characteristic function of Q . Then δ_Q is a bipolar complex fuzzy quasi-ideal. By (3), $\delta_Q = \delta_Q\Gamma M\Gamma\delta_Q\Gamma M\Gamma\delta_Q$.

Evaluating at a , since $\delta_Q(a) = 1$ in the positive components, the equality forces the existence of elements $x, y \in M$ and $\alpha, \beta \in \Gamma$ such that $a = aax\beta a$.

Hence $a \in a\Gamma M\Gamma a$, and therefore M is regular. Thus (1), (2), and (3) are equivalent.

3. Conclusion:

In this paper, the bipolar complex fuzzy quasi-ideals and bi-ideals in the context of γ -semirings were introduced and we established important results using these definitions. After laying out the basic setup, we checked how these fuzzy structures behave and proved a few results that follow from the way their positive and negative parts interact with the γ -operation. The study shows that this mixed fuzzy approach fits quite naturally with γ -semirings.

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