

NIL-CLEAN GRAPH OF SOME POLYNOMIAL RINGS

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Abstract: A nil-clean element of a commutative ring R is an element which can be written as a sum of an idempotent element and a nil element of R . The graph constructed by taking all nonzero elements of a ring as the vertices, where any two vertices are adjacent if their sum is a nil-clean element of R is known as nil-clean graph. With a revised adjacency rule, as any two vertices are adjacent if their product is a nil-clean element, on the same vertex set we revised the definition of nil-clean graph. In this work we study the Nil-Clean Graph of some polynomial quotient rings and their graph theoretical properties.

Keywords: Divisor, Prime factor, Greatest common divisor, Connectedness, Diameter, Girth, Radius, Isomorphism, Planar graph.

1. Introduction:

The exploration of algebraic structures via graphs has developed into a significant field of research. Beck initiated this by presenting the zero-divisor graph for commutative rings by defining the graph with all ring elements as vertices [3]. Later, Anderson and Livingston[1] refined it. They used only nonzero zero divisors as vertices. Two vertices are connected if their product is zero. So, for a zero-divisor graph $\Gamma(R)$ of a commutative ring R , $V(\Gamma(R)) = Z(R)^*$, the set of nonzero zero divisors of R , and any two distinct vertices x and y are adjacent in $\Gamma(R)$ if $xy = 0$ [1]. This idea showed strong links between ring and graph properties. Many graphs based on rings have since been studied.

A new step in ring theory came from clean and nil-clean decompositions, when Diesl [4] introduced nil-clean rings in 2013. The author characterized the rings in which every element can be written as the sum of an idempotent and a nilpotent element. Koşan and Zhou[8] gave generalizations as weakly nil-clean rings. This expanded the scope of these ideas by allowing decompositions of the form $x = n + e$ or $x = n - e$, where e is idempotent and n is nilpotent.

Inspired by the interplay between ring properties and graph structures, Sharma and Basnet [9] introduced the nil-clean divisor graph $G_N(R)$ associated with a finite commutative ring. In their formulation, the vertex set consists of all nonzero elements $x \in R$ for which some nonzero $y \in R$ satisfies $xy \in NC(R)$, the set of nil-clean elements of R . Two such vertices x and y are adjacent exactly when their product is nil-clean. Their work unified and generalized earlier constructions such as idempotent-divisor graphs and nilpotent-divisor graphs, and investigated several structural aspects including girth, clique number, connectedness, and domination parameters.

In this article, we focus on a revised idea by redefining the nil-clean graph $G_{NC}(R)$ of a commutative ring R with unity. In this graph, all nonzero elements of R act as vertices. Two vertices, x and y have an edge between them if the product xy is a nil-clean element of R . This changes the connections between the vertices, maintaining the connection to the nil-clean structure of the ring. The study of $G_{NC}(R)$ is another extension of earlier works, which shows new graph behaviours that closely relate to the properties of polynomial quotient rings.

Our research mainly focuses on polynomial quotient rings of the type $\frac{\mathbb{Z}_n[x]}{\langle f(x) \rangle}$, for small positive integers n and polynomials $f(x) \in \frac{\mathbb{Z}_n[x]}{\langle f(x) \rangle}$. These rings may range from finite fields to finite local or semilocal rings. They offer several instances for investigating changes in the structure of $G_{NC}(R)$ in various algebraic situations. If $f(x)$ is irreducible, then the ring $\frac{R[x]}{\langle f(x) \rangle}$ is a field. Then 0 and 1 are the only nil-clean elements in the ring. So, in the

nil-clean graph $G_{NC}(R)$, only the multiplicative inverses are adjacent to each other. That's why the graph just breaks down into simple pieces, groups of K_1 's and K_2 's, which is exactly what Theorem 3.1 shows. But things get more interesting with reducible polynomials, especially if you've got repeated roots. Suddenly, there are way more nil-clean elements in the ring. The graph gets a lot denser and more complicated. Our results lay out how this plays out for quadratic and higher-degree polynomials over $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$, and $\mathbb{Z}_p$.

The nil-clean graph shows how the shape of the graph shifts when you change things like whether a polynomial is irreducible or how many times a root repeats in a ring. The number of idempotent and nilpotent elements in a ring really decides if the graph is connected. Properties like connectedness, diameter, girth, and dominance number actually depend on these algebraic details. These graphs give us a new way to link ring theory with graph theory. Idempotent and nilpotent elements, for example, shape the graph in their own unique ways. These graphs let us see how the character of a ring shows up visually, in the structure of a graph. In our work, we dug into how these patterns show up in rings through their nil-clean elements.

2. Preliminaries:

In this section we provide definition of some important terminologies which are used in this study. We begin with some algebraic terms. An element e of a ring R is called idempotent if it satisfies the equation $e^2 = e$. An element $n \in R$ is nilpotent if there exists a positive integer k such that $n^k = 0$. An element $x \in R$ is nil-clean if it is expressible as $x = e + n$, sum of e , an idempotent, and n , a nilpotent element of R . A ring R is a nil-clean ring if all its elements are nil-clean, i.e., $NC(R) = R$. For a ring R and polynomial $f(x) \in R[x]$, the quotient ring $R[x]/\langle f(x) \rangle$ consists of all equivalence classes of polynomials modulo the ideal generated by $f(x)$. Every element can be represented by a polynomial of degree less than $\deg(f)$. A polynomial $f(x) \in \mathbb{Z}_p[x]$ is irreducible if it has no nontrivial factorisation; otherwise, it is reducible. If $f(x)$ is irreducible over $\mathbb{Z}_p[x]$, then the quotient ring is a field of order $p^{\deg(f)}$, and the only nil-clean elements are 0 and 1. A nonzero element $a \in R$ is a zero divisor if there exists $b(\neq 0) \in R$ such that $ab = 0$. A nonzero element $u \in R$ is a unit if it has a multiplicative inverse in R .

Following is some basic graph concepts used within the article.

A graph is *connected* if every pair of vertices is joined by at least one path. *Diameter* is the maximum distance between any two vertices. *Radius* is the minimum eccentricity among all vertices. *Girth* is length of the shortest cycle. *Clique* is a set of mutually adjacent vertices. *Dominating set* is a set whose vertices dominate the graph. *Complete graph* is every vertex is adjacent to every other, denoted by K_n . *Complete bipartite graph* is partitioned into two sets with all cross edges, denoted by $K_{m,n}$.

In this article discussed some graph theoretic properties of the ring $\frac{\mathbb{Z}_n[x]}{\langle f(x) \rangle}$ for some particular values of n .

Definition 2.1 (Nil-Clean Graph). Given a ring R , the nil-clean graph, $G_{NC}(R)$ is defined as:

- i) All nonzero elements of R are the vertices.
- ii) There is an edge between two vertices x and y are adjacent if $xy \in NC(R)$.

Some of the properties frequently used in nil-clean graphs

- In finite fields, $NC(R) = \{0,1\}$, so adjacency occurs only between inverse pairs.
- In rings with nontrivial idempotents, the nil-clean graph may contain complete or bipartite components.
- Quotients by polynomials like $(x + a)^2$ produce large nilpotent sets, yielding dense graph structures such as unions of $K_p, K_{p,p}$ etc.

3. Properties of Nil-Clean Graph of $G_{NC}\left(\frac{\mathbb{Z}_p[x]}{\langle f(x) \rangle}\right)$

Let us start the with the formal definition of the nil-clean graph $G_{NC}(R)$.

Definition 3.1. Let R be a commutative ring with unity. The nil-clean graph of a ring is a graph with all non-zero elements of the ring R as vertices and any two elements x and y are adjacent if $xy \in G_{NC}(R)$, the set of nil-clean elements of R .

Example 3.1. $G_{NC}\left(\frac{\mathbb{Z}_2[x]}{\langle x^4+x^3+x^2+x+1 \rangle}\right) \cong K_1 \cup 7K_2$ (Figure 1(a)) and $G_{NC}\left(\frac{\mathbb{Z}_3[x]}{\langle x^3+2x^2+1 \rangle}\right) \cong 2K_1 \cup 12K_2$ (Figure 1(b)).

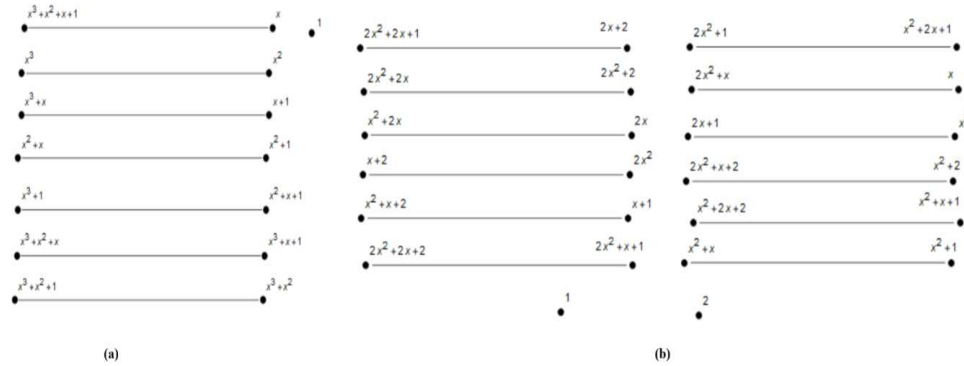


Figure 1: (a) $G_{NC}\left(\frac{\mathbb{Z}_2[x]}{\langle x^4+x^3+x^2+x+1 \rangle}\right)$ (b) $G_{NC}\left(\frac{\mathbb{Z}_3[x]}{\langle x^3+2x^2+1 \rangle}\right)$

Theorem 3.1. For all prime p and irreducible polynomial $f(x) \in \mathbb{Z}_p[x]$ of degree m , the following are true for $G_{NC}\left(\frac{\mathbb{Z}_p[x]}{\langle f(x) \rangle}\right)$

- $G_{NC}\left(\frac{\mathbb{Z}_p[x]}{\langle f(x) \rangle}\right)$ is disconnected.
- The graph $G_{NC}\left(\frac{\mathbb{Z}_p[x]}{\langle f(x) \rangle}\right)$ is a forest.
- If $p=2$, 1 is the only isolated vertex, and for all prime $p \neq 2$, the vertices 1 and $p-1$ are always isolated vertex.
- $G_{NC}\left(\frac{\mathbb{Z}_p[x]}{\langle f(x) \rangle}\right) \cong \begin{cases} K_1 \cup \left(\frac{p^m-2}{2}\right) K_2, & p=2 \\ 2K_1 \cup \left(\frac{p^m-3}{2}\right) K_2, & p \neq 2 \end{cases}$ where $m = \deg(f(x))$

Proof:

As $\frac{\mathbb{Z}_p[x]}{\langle f(x) \rangle}$ is a field, every vertex of $G_{NC}\left(\frac{\mathbb{Z}_p[x]}{\langle f(x) \rangle}\right)$ is adjacent only to its inverse. So $G_{NC}\left(\frac{\mathbb{Z}_p[x]}{\langle f(x) \rangle}\right)$ is disconnected.

As every vertex is adjacent only to its inverse only, the components of $G_{NC}\left(\frac{\mathbb{Z}_p[x]}{\langle f(x) \rangle}\right)$ are either K_1 or K_2 . So

$G_{NC}\left(\frac{\mathbb{Z}_p[x]}{\langle f(x) \rangle}\right)$ is a forest.

In $\frac{\mathbb{Z}_p[x]}{\langle f(x) \rangle}$, 1 is the only self-inverse element and 1 and $p-1$ both are only self-inverse elements in $\frac{\mathbb{Z}_p[x]}{\langle f(x) \rangle}$ if $p \neq 2$.

Being a simple graph $G_{NC}\left(\frac{\mathbb{Z}_p[x]}{\langle f(x) \rangle}\right)$, 1 and $p-1$ are isolated vertices in the graph.

Since $\left|V\left(G_{NC}\left(\frac{\mathbb{Z}_p[x]}{\langle f(x) \rangle}\right)\right)\right| = p^m - 1$, the number of K_2 component in the graph is $\frac{p^m-2}{2}$ if $p=2$ and $\frac{p^m-3}{2}$ if $p \neq 2$. Thus, the results follow.

Corollary 3.1.1. $Dom\left(G_{NC}\left(\frac{\mathbb{Z}_p[x]}{\langle f(x) \rangle}\right)\right) = \begin{cases} \frac{p^m-2}{2} + 1, & p=2 \\ \frac{p^m-3}{2} + 2, & p \neq 2 \end{cases}$, where p is prime and $f(x) \in \mathbb{Z}_p[x]$ is irreducible,

of degree m .

Proof: From Theorem 3.1, the result is a direct consequence.

Theorem 3.2. For all reducible quadratic polynomial $f(x) \in \mathbb{Z}_2[x]$, $G_{NC}\left(\frac{\mathbb{Z}_2[x]}{\langle f(x) \rangle}\right) \cong K_3$ (or $K_1 + K_2$).

Proof: The set of nil-clean elements of $\left(\frac{\mathbb{Z}_2[x]}{\langle f(x) \rangle}\right)$ is $\{0, 1, x, x + 1\} = \mathbb{Z}_2[x]$, i.e. the ring is a nil-clean ring and hence is complete graph with three vertices. Thus $G_{NC}\left(\frac{\mathbb{Z}_2[x]}{\langle f(x) \rangle}\right) \cong K_3$ (or $K_1 + K_2$).

Corollary 3.2.1. $Dom\left(G_{NC}\left(\frac{\mathbb{Z}_2[x]}{\langle f(x) \rangle}\right)\right) = 1$, where $f(x) \in \mathbb{Z}_2[x]$ is reducible quadratic polynomial.

Theorem 3.3. Let $f(x) \in \mathbb{Z}_2[x]$ be a reducible polynomial with all equal zeros. Then

$$G_{NC}\left(\frac{\mathbb{Z}_2[x]}{\langle f(x) \rangle}\right) \cong \begin{cases} 3K_2 + K_1 & \text{if } \deg(f(x)) = 3 \\ 3K_4 + K_3 & \text{if } \deg(f(x)) = 4 \\ (2K_4 \cup K_{4,4} \cup (K_4 + 8K_1)) + K_3 & \text{if } \deg(f(x)) = 5 \\ (2K_8 \cup K_{8,8} \cup (K_8 + 16K_1)) + K_7 & \text{if } \deg(f(x)) = 6 \end{cases}$$

Theorem 3.4. For all reducible n th degree polynomial $f(x) \in \mathbb{Z}_2[x]$ with all equal zeros

$$G_{NC}\left(\frac{\mathbb{Z}_2[x]}{\langle f(x) \rangle}\right) \cong \left(2K_{\lfloor \frac{n}{2} \rfloor} \cup K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor} \cup \left(K_{\lfloor \frac{n}{2} \rfloor} + \left(2^{\lfloor \frac{n}{2} \rfloor + 1}\right) K_1\right)\right) + K_{\lfloor \frac{n}{2} \rfloor - 1}$$

Corollary 3.4.1. For all reducible n th degree polynomial $f(x) \in \mathbb{Z}_2[x]$ with all equal zeros,

$$Dom\left(G_{NC}\left(\frac{\mathbb{Z}_2[x]}{\langle f(x) \rangle}\right)\right) = 2^{\lfloor \frac{n}{2} \rfloor} - 1.$$

Theorem 3.5. For all $a \in \mathbb{Z}_3$, $G_{NC}\left(\frac{\mathbb{Z}_3[x]}{\langle (x+a)^2 \rangle}\right) \cong 2K_3 + K_2$.

Proof: The set of nil-clean elements of $\frac{\mathbb{Z}_3[x]}{\langle (x+a)^2 \rangle}$ is $\{0, 1, x + a, 2(x + a), (x + a) + 1, 2(x + a) + 1\}$ and the vertex set of $G_{NC}\left(\frac{\mathbb{Z}_3[x]}{\langle (x+a)^2 \rangle}\right)$ are $\{1, 2, x + a, 2(x + a), (x + a) + 1, (x + a) + 2, 2(x + a) + 1, 2(x + a) + 2\}$. Each of the sets $\{1, (x + a) + 1, 2(x + a) + 1\}$, and $\{2, (x + a) + 2, 2(x + a) + 2\}$ gives a complete subgraph K_3 and the set $\{x + a, 2(x + a)\}$ gives a complete subgraph K_2 . The elements of the sets $\{1, (x + a) + 1, 2(x + a) + 1\}$ and $\{2, (x + a) + 2, 2(x + a) + 2\}$ are not adjacent to each other but are adjacent to every element of the set $\{x + a, 2(x + a)\}$ which give us the graph isomorphic to $2K_3 + K_2$.

Theorem 3.6. For all $a \in \mathbb{Z}_5$, $G_{NC}\left(\frac{\mathbb{Z}_5[x]}{\langle (x+a)^2 \rangle}\right) \cong (2K_5 \cup K_{5,5}) + K_4$.

Theorem 3.7. For all $a \in \mathbb{Z}_p$, $G_{NC}\left(\frac{\mathbb{Z}_p[x]}{\langle (x+a)^2 \rangle}\right) \cong (2K_p \cup \frac{p-3}{2} K_{p,p}) + K_{p-1}$.

Corollary 3.7.1. For all $a \in \mathbb{Z}_p$, $Dom\left(G_{NC}\left(\frac{\mathbb{Z}_p[x]}{\langle (x+a)^2 \rangle}\right)\right) = p - 1$.

Proof: From Theorem 3.7, the result is a direct consequence.

Theorem 3.8. For all $a, b \in \mathbb{Z}_3$, $a \neq b$, $G_{NC}\left(\frac{\mathbb{Z}_3[x]}{\langle (x+a)(x+b) \rangle}\right) \cong 4K_3$.

Proof: The nil-clean elements of $\frac{\mathbb{Z}_3[x]}{\langle (x+a)(x+b) \rangle}$ are $0, 1, 2(x + a), x + b$. The elements of each of the sets $\{1, 2(x + a), x + b\}$, $\{2, (x + a), 2(x + b)\}$, $\{(x + a) + 2, 2(x + a), 2(x + b)\}$ and $\{(x + b) + 2, (x + a), (x + b)\}$ forms a K_3 . Also $(x + a) - (x + b) - 2(x + a) - 2(x + b) - (x + a)$ forms a cycle. Thus, the graph $G_{NC}\left(\frac{\mathbb{Z}_3[x]}{\langle (x+a)(x+b) \rangle}\right)$ has four vertices of degree 2 and four vertices of degree 4 as shown in figure 3(b).

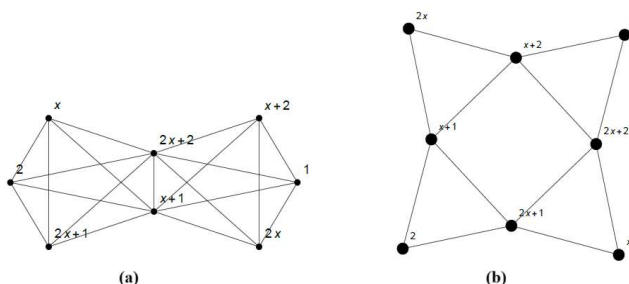


Figure 2. (a) $G_{NC}\left(\frac{\mathbb{Z}_3[x]}{\langle x^2+2x+1 \rangle}\right)$ (b) $G_{NC}\left(\frac{\mathbb{Z}_3[x]}{\langle x^2+2 \rangle}\right)$

Theorem 3.9. For all $a, b \in \mathbb{Z}_3$, $a \neq b$, $G_{NC}\left(\frac{\mathbb{Z}_p[x]}{\langle (x+a)(x+b) \rangle}\right) \cong 4K_3$ has the following properties

- i) There are exactly four vertices of degree 2.
- ii) There are exactly four vertices of degree $2p - 2$.
- iii) There are exactly $2(p-3)$ vertices of degree $2p - 1$.
- iv) There are exactly $(p-3)(p+1)$ vertices of degree 3.

Proof: For the ring $\frac{\mathbb{Z}_p[x]}{\langle (x+a)(x+b) \rangle}$, the nil-clean elements are $0, 1, x + b, (p-1)x + (p-1)a$.

Vertex 1 is adjacent to the vertices $x + b$ and $(p-1)x + (p-1)a$ only. Vertex $p-1$ is adjacent to the vertices $x + a$ and $(p-1)x + (p-1)b$ only. Vertex $3(x+a) + (p-1)$ is adjacent only to the vertices $x + b$ and $(p-1)x + (p-1)a$. Vertex $2(x+b) + (p-1)$ is adjacent only to the vertices $x + a$ and $(p-1)x + (p-1)b$. Thus, the vertices $1, p-1, 3(x+a) + (p-1), 2(x+b) + (p-1)$ are of degree 2 and these are only vertices in $G_{NC}\left(\frac{\mathbb{Z}_p[x]}{\langle (x+a)(x+b) \rangle}\right)$ with degree 2.

The four vertices of degree $2p - 2$ are $x + a, x + b, (p-1)x + (p-1)a, (p-1)x + (p-1)b$.

The $2(p-3)$ vertices of degree $2p - 1$ are of the form $k(x+a)$ and $k(x+b), k = 2, 3, \dots, (p-2)$.

Remaining $(p-3)(p+1)$ vertices are of degree 3.

Theorem 3.10. For all irreducible $f(x)$, $G_{NC}\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right) \cong (K_4 \cup K_{4,4}) + K_3$.

Proof: For irreducible $f(x)$, $N\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right) = \{0, 2, 2x, 2x+2\}$ and $I\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right) = \{0, 1\}$. So $N\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right) = \{0, 1, 2, 3, 2x, 2x+1, 2x+2, 2x+3\}$. The elements $2, 2x, 2x+2$ are adjacent to each other. Also, the elements in the set $\{1, 3, 2x+1, 2x+3\}$ form a complete graph. On the other hand, the elements of the set $\{x, 3x, x+2, 3x+2\} \cup \{x+1, x+3, 3x+3, 3x+1\}$ give a complete bipartite graph $K_{4,4}$. Now the elements of sets $\{1, 3, 2x+1, 2x+3\}$ and $\{x, 3x, x+2, 3x+2\} \cup \{x+1, x+3, 3x+3, 3x+1\}$ are not adjacent to each other but all are adjacent to the elements of the set $\{2, 2x, 2x+2\}$, which gives that $G_{NC}\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right) \cong (K_4 \cup K_{4,4}) + K_3$ (as in figure 4).

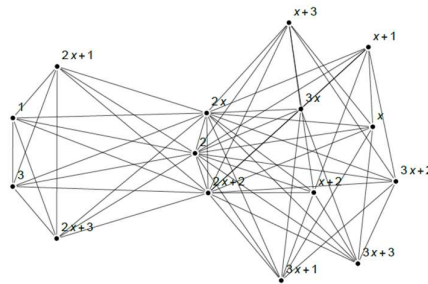


Figure 3: $G_{NC}\left(\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1 \rangle}\right)$

Theorem 3.11. Let $f(x) \in \mathbb{Z}_4[x]$ be any quadratic monic polynomial and $a, b \in \mathbb{Z}_4$ are the zeros of $f(x)$, then the following statements are true for $G_{NC}\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right)$

$G_{NC}\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right)$ is complete graph and is isomorphic to K_{15} if either $a = b$ or one of a or b is 2.

$G_{NC}\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right)$ is isomorphic to $3K_4 + K_3$ if $a \neq b \neq 2$.

Proof: When $a = b$, $NC\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right) = \frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}$. Thus, every pair of vertices is adjacent in $G_{NC}\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right)$ and the graph is complete. Since $\left|V\left(G_{NC}\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right)\right)\right| = 15$, $G_{NC}\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right) \cong K_{15}$.

Now let $a \neq b$ and let $a = 2$ then $N\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right) = \{0, 2, 2x, 2x+2\}$ and $I\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right) = \{0, 1, b(x+2), 3(x+b)\}$ which gives $NC\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right) = \frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}$. Therefore, as in previous case, we have $G_{NC}\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right) \cong K_{15}$.

Let $a \neq b \neq 2$, then $N\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right) = \{0, 2, 2x, 2x+2\}$ and $I\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right) = \{0, 1\}$. So $NC\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right) = \{0, 1, 2, 3, 2x, 2x+1, 2x+2, 2x+3\}$. In the graph $G_{NC}\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right)$, the vertices $2, 2x, 2x+2$ are adjacent to each other and gives K_3 . On the other hand, the elements of each of the sets $\{1, 3, 2x+1, 2x+3\}$, $\{x, 3x, x+2, 3x+2\}$ and $\{x+1, x+3, 3x+3, 3x+1\}$ gives us complete subgraphs K_4 . The elements of each of these three sets are not adjacent to that of the others but they all are adjacent to every element of the set $\{2, 2x, 2x+2\}$. This finally gives us the graph isomorphic to $3K_4 + K_3$. (as shown in figure 4).

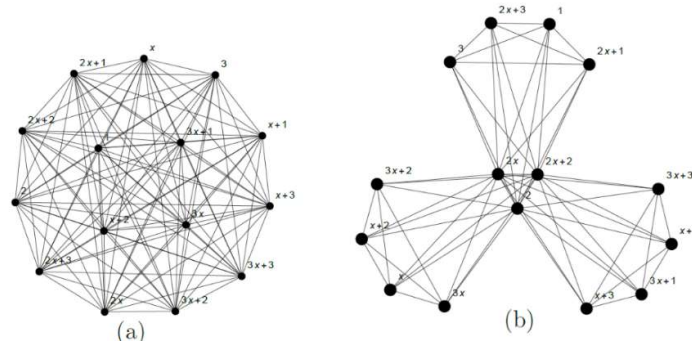


Figure 4: (a) $G_{NC}\left(\frac{\mathbb{Z}_4[x]}{\langle x^2+x+2 \rangle}\right)$ (b) $G_{NC}\left(\frac{\mathbb{Z}_4[x]}{\langle x^2+3x+2 \rangle}\right)$

Theorem 3.12. Let $f(x) \in \mathbb{Z}_4[x]$ be any quadratic monic polynomial and $a, b \in \mathbb{Z}_4$ are the zeros of $f(x)$, then the following statements are true for $G_{NC}\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right)$

- $f(x)$ is irreducible if $G_{NC}\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right) \cong (K_4 \cup K_{4,4}) + K_3$.
- $f(x)$ is irreducible if $G_{NC}\left(\frac{\mathbb{Z}_4[x]}{\langle f(x) \rangle}\right) \cong K_{15}$ or $3K_4 + K_3$.

4. Conclusion:

In this article the authors have studied nil-clean graphs of polynomial quotient rings $\frac{\mathbb{Z}_n[x]}{\langle f(x) \rangle}$ for some small values of n . The study shows that the connectedness of the graph not only depends on irreducibility of $f(x)$ but also on the multiplicity of the zeroes and the nil-clean elements of the rings. This also leads to variations in the values like domination number, diameter and radius of the graph. The authors have studied the graphs only for some small values of n . The work still remained open for the bigger values of n with complete prime factorization.

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