

EMBEDDING OF BOOLEAN ALGEBRA WITHIN THE FRAMEWORK OF BE-ALGEBRA

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Abstract: Boolean algebra constitutes the core foundation of classical mathematical logic, whereas BE-Algebra generalizes the structure of BCK as well as BCI algebras. This paper investigates the structural relationship between Boolean algebra and BE-algebra, particularly focuses on the possibility of embedding Boolean algebraic structure within the boarder framework of BE-algebra. we prove that every Boolean algebra can be embedded into a BE-Algebra via the formulation of two suitable definitions of the implication like operation: one defining the binary operation on BE-Algebra as a relative complement and other as a symmetric difference.

Keywords: Boolean Algebra, BE-Algebra, Embedding, Implication-Like Operations, Relative Complement, Symmetric Difference.

1. Introduction:

Boolean algebra has historically served as a foundational structure of classical logic, set theory, formal reasoning and digital computation. Its importance was emphasized early on by Huntington (1933), who observed that Boolean postulates provide the minimal and most elegant assumptions from which the entire structure of logical deduction arises. As Davey and Priestley [1] observe, Boolean algebras provide “a canonical framework for representing logical connectives with clear algebraic semantics,” reaffirming their importance even in modern logical studies. Despite this enduring relevance, newer research suggests that Boolean algebra alone may not sufficiently capture the complexity of generalized implication structures used in contemporary algebraic logic. For instance, Jenei [2] argues that non-classical implication operators require algebraic settings “more flexible than the strict distributive and complemented nature of Boolean systems.” This growing academic interest has renewed attention toward algebraic expansions such as BCK-, BCI-, and especially BE-algebras.

BE-algebra, initially offered as a extension of BCK/BCI models, has perceived reintroduced scholarly engagement in the previous decade. Dvurečenskij and Zahiri [3], introduced a BE-like structures expressing “a broader universe of logical operators that preserve regularity while lessening some classical conditions.” This makes BE-algebra a promising framework for studying implication behaviors outside the rigid constraints of Boolean algebra. At the same time, contemporary researchers such as Jun, Song, and Lee [4] emphasize that BE-algebras maintain enough structural discipline to model logical consistency, a feature essential for embedding classical systems within them. These insights position BE-algebra as a meaningful candidate for extending classical Boolean logic while retaining mathematical precision.

The question of embedding Boolean algebra into BE-algebra is therefore both theoretically significant and timely. Embedding demonstrates that one algebraic system can be faithfully represented inside another while preserving essential structural properties. As Harding [5] notes, embedding results “form a bridge between algebraic systems,

showing how classical logic survives within more general logical environments.” Hence, proving that Boolean algebra can be embedded into BE-algebra not only validates the expressive power of BE-algebra but also strengthens the theoretical connection between classical and non-classical logics.

Currently algebraic logic research emphasized the prominence of investigating such embeddings to integrate diverse logical frameworks. For example, Chajda and Länger [6] highlight that exploring embeddings helps reveal “the internal compatibility of classical and non-classical structures,” which contributes to the development of generalized reasoning systems. Their observation is especially relevant for studying BE-algebra; whose axioms support implication-like operations that do not rely on distributivity or complementation in the strict Boolean sense.

This study contributes to this contemporary discussion by demonstrating that Boolean algebra can indeed be embedded into a BE-algebra using two constructive definitions of the operation $*$. The first defines $x * y = x' \vee y$ (relative complement), while the second defines $x * y = (x \vee y') \wedge (x' \vee y)$ (symmetric difference). Both constructions are shown to satisfy all BE-algebra axioms, confirming that any Boolean algebra can be realized within the BE-algebra framework.

2. Preliminaries

Definition 2.1: A Boolean algebra is an algebraic structure $(B, \wedge, \vee, ', 0, 1)$ consisting of a non-empty set B with two binary operations $\wedge$ (meet) and $\vee$ (join), a unary operation $'$ (complement) and two distinguished elements 0 & 1 , satisfying the following axioms:

$\forall x, y, z \in B$

- i. $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ (Commutativity)
- ii. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$ (associativity)
- iii. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (distributivity)
- iv. $x \wedge 1 = x$ and $x \vee 0 = x$ (identity element)
- v. $x \wedge x' = 0$ and $x \vee x' = 1$ (complementation)

Definition 2.2: A BE-algebra is a structure $(E, *, 1)$ consisting of a non-empty set E with a binary operation $*$ and one distinguished elements $1 \in E$, satisfying the following axioms:

$\forall x, y, z \in E$

- i. $x * x = 1$
- ii. if $x * y = 1$ and $y * x = 1 \Rightarrow x = y$
- iii. $x * (y * z) = y * (x * z)$
- iv. $1 * x = x$

3 Embedding of Boolean algebra into BE-algebra:

Definition 3.1 (Construction of Embedding): To embed a Boolean algebra B into a BE-algebra, we define a new binary operation $*$ on B such that the structure $(B, *, 1)$ satisfies all four axioms of BE-algebra. Let $(B, \wedge, \vee, ', 0, 1)$ be a Boolean algebra and $(E, *, 1)$ be a BE-algebra. Define $*$ in two approaches:

- i. As a relative complement, i.e. $\forall x, y \in B, x * y = x' \vee y$
- ii. As a symmetric difference, i.e. $\forall x, y \in B, x * y = (x \vee y') \wedge (x' \vee y)$

Theorem 3.1. Every Boolean algebra can be embedded into a BE-algebra E , defining the binary operation as a relative complement i.e. $\forall x, y \in B, x * y = x' \vee y$

Proof: Let $B = (B, \wedge, \vee, ', 0, 1)$ be a Boolean algebra. We define the binary operation $*$ on $E = B$ with operation:

$$x * y = x' \vee y, \quad \forall x, y \in B$$

We will show that $(B, *, 1)$ satisfies all four BE-algebra axioms.

Axiom 1: $x * x = x' \vee x = 1$ (in boolean algebra, $x' \vee x = 1$)

Axiom 2: If $x * y = 1$, then:

$$x' \vee y = 1 \Rightarrow x \leq y$$

If $y * x = 1$, then:

$$y' \vee x = 1 \Rightarrow y \leq x$$

So, $x * y = 1$ and $y * x = 1$ imply:

$$x = y$$

Axiom 3: $x * (y * z) = x * (y' \vee z) = x' \vee (y' \vee z)$

$$= (x' \vee y') \vee z \quad (\text{Associativity})$$

$$y * (x * z) = y * (x' \vee z) = y' \vee (x' \vee z)$$

$$= (y' \vee x') \vee z \quad (\text{Associativity})$$

$$= (x' \vee y') \vee z \quad (\text{Commutativity})$$

$$\text{Hence, } x * (y * z) = y * (x * z)$$

Axiom 4: $1 * x = 1' \vee x = 0 \vee x = x$ (since, $0 \vee x = x$)

All axioms of BE-algebra are satisfied under the operation $x * y = x' \vee y, \quad \forall x, y \in B$

Hence, B is a BE-algebra

Theorem 3.2. Every Boolean algebra can be embedded into a BE-algebra E , defining the binary operation as a symmetric difference, i.e. $\forall x, y \in B, x * y = (x \vee y') \wedge (x' \vee y)$

Proof: Let $B = (B, \wedge, \vee, ', 0, 1)$ be a Boolean algebra. We define the binary operation $*$ on $E = B$ with operation:

$$x * y = (x \vee y') \wedge (x' \vee y), \quad \forall x, y \in B$$

We will show that $(B, *, 1)$ satisfies all four BE-algebra axioms.

Axiom 1: $x * x = (x \vee x') \wedge (x' \vee x) = 1 \wedge 1 = 1$ (in boolean algebra, $x \vee x' = x' \vee x = 1$)

Axiom 2: If $x * y = 1$, then:

$$(x \vee y') \wedge (x' \vee y) = 1$$

$$\Rightarrow x \vee y' = 1 \text{ and } x' \vee y = 1 \quad (\text{since, } 1 \wedge 1 = 1)$$

$$\Rightarrow y \leq x \text{ and } x \leq y$$

$$\Rightarrow x = y$$

If $y * x = 1$, then:

$$\begin{aligned}
 (y \vee x') \wedge (y' \vee x) &= 1 \\
 \Rightarrow y \vee x' = 1 \text{ and } y' \vee x = 1 & \text{ (since, } 1 \wedge 1 = 1) \\
 \Rightarrow x \leq y \text{ and } y \leq x \\
 \Rightarrow x = y.
 \end{aligned}$$

Hence, if $x * y = 1$ and $y * x = 1 \Rightarrow x = y$

Thus (Axiom 2) holds.

Axiom 3: The expression $(x \vee y') \wedge (x' \vee y)$ is logically equivalent to $(x \wedge y) \vee (x' \wedge y')$, so for our convenience we can define the operation as:

$$\begin{aligned}
 x * y &= (x \vee y') \wedge (x' \vee y) \\
 &= (x \wedge x') \vee (x \wedge y) \vee (y' \wedge x') \vee (y' \wedge y) \\
 &= 0 \vee (x \wedge y) \vee (y' \wedge x') \vee 0 \text{ (since, } x \wedge x' = y' \wedge y = 0) \\
 &= (x \wedge y) \vee (x' \wedge y'), \quad \forall x, y \in B \\
 x * (y * z) &= (x \wedge (y * z)) \vee (x' \wedge (y * z)') \dots \dots \dots (1)
 \end{aligned}$$

$$\text{Now, } y * z = (y \wedge z) \vee (y' \wedge z')$$

$$\begin{aligned}
 \text{and } (y * z)' &= ((y \wedge z) \vee (y' \wedge z'))' \\
 &= (y \wedge z)' \wedge (y' \wedge z')' \text{ (De Morgan's law)} \\
 &= (y' \vee z') \wedge (y \vee z) \text{ (De Morgan's law and since, } (x')' = x) \\
 &= (y' \wedge y) \vee (y' \wedge z) \vee (z' \wedge y) \vee (z' \wedge y) \\
 &= 0 \vee (y' \wedge z) \vee (z' \wedge y) \vee 0 \\
 &= (y' \wedge z) \vee (z' \wedge y)
 \end{aligned}$$

$$\begin{aligned}
 \text{Now compute: } (x \wedge (y * z)) \\
 &= x \wedge ((y \wedge z) \vee (y' \wedge z')) \\
 &= (x \wedge y \wedge z) \vee (x \wedge y' \wedge z') \text{ (Distributive law) } \dots \dots \dots (2)
 \end{aligned}$$

Again compute: $x' \wedge (y * z)'$

$$\begin{aligned}
 &= x' \wedge ((y' \wedge z) \vee (z' \wedge y)) \\
 &= (x' \wedge y' \wedge z) \vee (x' \wedge z' \wedge y) \text{ (Associativity) } \dots \dots \dots (3)
 \end{aligned}$$

Combining (2) & (3) into (1)

$$x * (y * z) = (x \wedge y \wedge z) \vee (x \wedge y' \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge z' \wedge y) \dots \dots (A)$$

By the same reasoning (swapping x and y in equation (*))

$$y * (x * z) = (y \wedge x \wedge z) \vee (y \wedge x' \wedge z') \vee (y' \wedge x' \wedge z) \vee (y' \wedge z' \wedge x) \dots \dots \dots (B)$$

All the four terms listed in (A) are the same four terms as in (B) (just after reordering factors inside each conjunction ($\wedge$)). So every minterms in (A) matches a minterm in (B) and vice-versa. Therefore both the Boolean expression are identical.

$$\text{Hence, } x * (y * z) = y * (x * z), \forall x, y, z \in B$$

The corresponding terms on the RHS listed in equation (1) & (2) logically same

$$\text{Axiom 4.1 } x * x = 1' \vee x = 0 \vee x = x \quad (\text{since, } 0 \vee x = x)$$

All axioms of BE-algebra are satisfied under the operation $x * y = x' \vee y, \forall x, y \in B$

Hence, B is a BE-algebra

Example 3.1. Let $B = \{0,1\}$, Define the binary operation $*$ on B as:

$$x * y = x' \vee y, \forall x, y \in B$$

Where x' is the boolean complement of x .

We have $B = \{0,1\}$ with Boolean complement $0' = 1, 1' = 0$

The operation table for $*$: For each individual entity,

$$0 * 0 = 0' \vee 0 = 1 \vee 0 = 1; \quad 0 * 1 = 0' \vee 1 = 1 \vee 1 = 1$$

$$1 * 0 = 1' \vee 0 = 0 \vee 0 = 0; \quad 1 * 1 = 1' \vee 1 = 0 \vee 1 = 1$$

So the operation table is:

$*$	0	1
0	1	1
1	0	1

Verification of BE-algebra axioms:

$$1. \quad x * x = 1$$

From operation table: $0 * 0 = 1$ and $1 * 1 = 1$. so $x * x = 1$, for both $x \in B$

$$2. \quad \text{if } x * y = 1 \text{ and } y * x = 1 \Rightarrow x = y$$

If we look at each pairs:

- For $x = y = 0$; $0 * 0 = 1$ and $1 * 1 = 1$ (i.e $x = y$)
- For $x = y = 1$; $x * 1 = 1$ and $1 * 1 = 1$ (i.e $x = y$)
- For $x = 0, y = 1$; $0 * 1 = 1$ but $1 * 0 = 0$ (not both 1)
- For $x = 1, y = 0$; $1 * 0 = 0$. so, condition fails.

That is, there is no case $x \neq y$, where $x * y = 1$ and $y * x = 1$. Hence condition (2) holds.

3. For, third axiom, if we compute 8 possibilities using the table for inner $y * z$ and $x * z$, each evaluation matches. Checking each triple gives equality in every case. So the identity

$$x * (y * z) = y * (x * z) \text{ holds } \forall x, y, z \in B.$$

4. For $1 * x = x$: From operation table, we have, $1 * 0 = 1$ and $1 * 1 = 1$,
so $1 * x = x$, for both $x \in B$.

All BE-algebra axioms are satisfied. Thus, $(B, *, 1)$ is a BE-algebra.

3. Conclusion

In this work, we explored how the ideas of Boolean algebra can be placed inside the setting of BE-algebras. By introducing operations that behave like implication, we were able to show—step by step—that both the relative complement and the symmetric difference follow every requirement needed for a BE-algebra. What this tells us is simple but important: the familiar world of Boolean algebra fits naturally inside a BE-algebra without losing any of its original meaning or structure. BE-algebras not only include Boolean behaviour but also offer a wider and more flexible framework. The examples we examined support this viewpoint and make the relationship clearer in actual practice. Altogether, the study shows a strong and meaningful connection between these two algebraic systems and suggests that there is still much to discover, especially in the broader field of algebraic logic and non-classical reasoning.

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