

NEW INFINITE FAMILIES OF CONGRUENCE MODULO POWERS OF 2 OF OVERCUBIC PARTITION WITH 5 and 6– TUPLES

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Abstract: The overcubic partition of a positive integer n with k – tuples is denoted $\bar{b}_k(n)$. Our aim is to investigate the congruence behaviour of $\bar{b}_k(n)$. We will explore a number of Ramanujan-Type congruences and investigate some infinite families of congruence modulo powers of 2 of $\bar{b}_k(n)$ for $k = 5, 6$.

Keywords: Overcubic partition, Dissections, Congruences, Generating Function

1. Introduction:

A partition of an integer $n \geq 0$ is a non-increasing sequence of $\delta_1, \delta_2, \delta_3 \dots \dots \delta_m$ with $\delta_i \geq 0$ for $i = 1, 2, 3 \dots m$ so that

$$\sum_{i=1}^m \delta_i = n$$

δ_i knows as parts. It is denoted by $p(n)$ and $p(0) = 1$.

Ramanujan [18, 19] studied the arithmetic properties of $p(n)$ and established the following three congruences

$$p(5n + 4) \equiv 0 \pmod{5}$$

$$p(7n + 5) \equiv 0 \pmod{7}$$

$$p(11n + 6) \equiv 0 \pmod{11}$$

The generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)} = \frac{1}{(q; q)_{\infty}} = \frac{1}{f_1}$$

where

$$f_k = (q^k; q^k)_\infty = \prod_{n=1}^{\infty} (1 - q^{kn}),$$

where q is a complex number with $|q| < 1$ and $k < 1$.

According to Corteel and Lovejoy [20], overpartition of an integer $n \geq 0$ is a partition of n with the restriction that first occurrence of its part may be overlined. It is represented as $\bar{p}(n)$ and its generating function is defined as

$$\sum_{n=0}^{\infty} \bar{p}(n) q^n = \prod_{n=1}^{\infty} \frac{1 + q^n}{1 - q^n} = \frac{f_2}{f_1^2}$$

A cubic partition of an integer $n \geq 0$ is a partition where even parts can appear in two colors. It is denoted by $a(n)$ with $a(0) = 1$.

In 2010, H. C. Chan [3, 4, 6] introduced the cubic partition function $a(n)$. The generating function of $a(n)$ is given by

$$\sum_{n=0}^{\infty} a(n) q^n = \frac{1}{(q; q)_\infty (q^2; q^2)_\infty} = \frac{1}{f_1 f_2}$$

H. C. Chan, B. Kim [10] defined the overcubic partition function $\bar{a}(n)$ and the generating function for $\bar{a}(n)$ is given by

$$\sum_{n=0}^{\infty} \bar{a}(n) q^n = \frac{f_4}{f_1^2 f_2}$$

The number of cubic partition pair denoted by $b(n)$ was studied by Zhao and Zhong [16]. The generating function for $b(n)$ is given by

$$\sum_{n=0}^{\infty} b(n) q^n = \frac{1}{f_1^2 f_2^2}$$

Zhao and Zhong [16] explored certain arithmetic behaviour of $b(n)$.

The overcubic partition of n , denoted by $\bar{b}(n)$ was studied by Kim [9] and Kim defined the overcubic partition as the number of 4-color partitions of n where the colors are taken as r, y, o , and b so that the even parts only can have colors o and b and the first occurrence parts of the partition may be overline. The generating function of $\bar{b}(n)$ is defined as follows

$$\sum_{n=0}^{\infty} \bar{b}(n) q^n = \frac{f_4^2}{f_1^4 f_2^2}$$

If $\lambda_i, i = 1, 2, 3, \dots, k$ are overcubic partition of a positive integer n with $\sum_{i=1}^k \lambda_i = n$ then the k -tuple $(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k)$ is called overcubic partition of n with k -tuples. It is denoted by $\bar{b}_k(n)$ and the generating function is defined as

$$\sum_{n=0}^{\infty} \bar{b}_k(n) q^n = \frac{f_4^k}{f_1^{2k} f_2^k}$$

Several congruences modulo 3 and 64 for $\bar{b}_2(n)$ was established by Kim [9]. Motivated by Kim's work, Lin [11] proved many infinite families of congruence modulo 3 and 5.

Congruence modulo power of 2 for $\bar{b}_2(n)$ was studied by Ray and Barman [14]. The authors able to establish Several Ramanujan-type congruence and infinite families of congruence module power of 2. Naika and Shivashankar [12] also studied the arithmetic properties of overcubic partition and established many infinite families of congruences modulo powers of 2. Recently, in 2024, Buragohain and Saikia [2] explored the congruence properties of $\bar{b}_{4m+2}(n)$, $\bar{b}_{4m+3}(n)$, $\bar{b}_{8m+2}$, $\bar{b}_{8m+5}(n)$ and $\bar{b}_{16m+4}(n)$ for integer $m \geq 1$.

2. Preliminaries

Ramanujan's general theta function is defined by

$$f(a, b) = \sum_{n=0}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1 \quad (2.1)$$

$$f(a, b) = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty} \quad (2.2)$$

(2.2) is obtained using Jacobi's Triple product identity.

Three special cases of $f(a, b)$ are defined for $|q| < 1$, by [[1], p.36, Entry 22]

$$\phi(q) = f(q, q) = \sum_{k=0}^{\infty} q^{k^2} = \frac{f_2^5}{f_1^2 f_4^2} \quad (2.3)$$

$$\psi(q) = f(q, q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{f_2^2}{f_1} \quad (2.4)$$

$$f(-q) = f(-q, -q^2) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2} = f_1 \quad (2.5)$$

We also define the following

$$\varphi(-q) = \frac{(q; q)_{\infty}^2}{(q^2; q^2)_{\infty}} \text{ and } \psi(-q) = \frac{(q; q)_{\infty} (q^4; q^4)_{\infty}}{(q^2; q^2)_{\infty}} \quad (2.6)$$

We will use the following dissection formulas and some important, already established results in proving our main results.

Lemma 2.1. *The following 2-dissections holds*

$$f_1^2 = \frac{f_2 f_8^5}{f_4 f_{16}^2} - 2q \frac{f_2 f_{16}^2}{f_8} \quad (2.8)$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} + 4q \frac{f_2^2 f_8^4}{f_4^2} \quad (2.9)$$

Proof. For (2,8) see [1, p. 40, Entry 25] and see [22, Lem. 1] for a proof of (2.9).

Lemma 2.2 ([17], Lemma. 2.1). *The following 2-dissections hold*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8} \quad (2.10)$$

$$\frac{1}{f_1^8} = \frac{f_4^{28}}{f_2^{28} f_8^8} + 8q \frac{f_4^{16}}{f_2^{24}} + 16q^2 \frac{f_4^4 f_8^8}{f_2^{20}} \quad (2.11)$$

$$\frac{1}{f_1^{12}} = \frac{f_4^{42}}{f_2^{42} f_8^{12}} + 12q \frac{f_4^{30}}{f_2^{38} f_8^4} + 48q^2 \frac{f_4^{18} f_8^4}{f_2^{34}} + 64q^3 \frac{f_4^6 f_8^{12}}{f_2^{30}} \quad (2.12)$$

$$\frac{1}{f_1^{16}} = \frac{f_4^{56}}{f_2^{56} f_8^{16}} + 16q \frac{f_4^{44}}{f_2^{52} f_8^8} + 96q^2 \frac{f_4^{32}}{f_2^{48}} + 256q^3 \frac{f_4^{20} f_8^8}{f_2^{44}} + 256q^4 \frac{f_4^8 f_8^{16}}{f_2^{40}} \quad (2.13)$$

Lemma 2.3 ([21], Equation 14.3.3). *We have, the following 3-dissection*

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9} \quad (2.14)$$

Lemma 2.4 ([1], p, 345). *The following 3-dissections holds:*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} + 4q^3 f_3^2 f_{18}^6 - 3q f_9^3 \quad (2.15)$$

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2 \frac{f_3 f_{18}^4}{f_6 f_9^2} \quad (2.16)$$

Lemma 2.5 ([7], Theorem 2.1). *For any odd prime p , we have*

$$\psi(q) = \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}} \psi(q^{p^2})$$

Moreover, for $0 \leq k \leq \frac{p-3}{2}$

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod{8}$$

Lemma 2.6 ([7], Theorem 2.2). *For any prime $p \geq 5$, we have*

$$f_1 = \sum_{\substack{k=-(p-1)/2 \\ k \neq \pm(p-1)/6}}^{(p-1)/2} (-1)^k q^{(3k^2+k)/2} f(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}$$

.where

$$\frac{\pm p-1}{6} = \begin{cases} \frac{p-1}{6} & \text{if } p \equiv 1 \pmod{6} \\ -\frac{p-1}{6} & \text{if } p \equiv -1 \pmod{6} \end{cases}$$

Furthermore, if

$$-\frac{p-1}{6} \leq k \leq \frac{p-1}{6} \quad \text{and } k \neq \frac{\pm p-1}{6}$$

then

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}$$

Lemma 2.7 ([5], Lemma 2.3). *For any prime $p \geq 3$, we have*

$$f_1^3 = \sum_{\substack{k=0 \\ k \neq \frac{p-1}{2}}}^{p-1} (-1)^k q^{\frac{k(k+1)}{2}} \sum_{n=0}^{\infty} (-1)^n (2pn+2k+1) q^{pn \frac{pn+2k+1}{2}} + (-1)^{p-1/2} q^{\frac{p^2-1}{8}} f_{p^2}^3$$

Furthermore, if $k \neq \frac{p-1}{2}$ and $0 \leq k \leq p-1$, then

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}$$

Lemma 2.8 ([13], p. 212). *We have the following 5-dissection formula:*

$$f_1 = f_{25} \left(a(q^5) - q - \frac{q^2}{a(q^5)} \right),$$

Where

$$a := a(q) := \frac{(q^2, q^3, q^5; q^5)_\infty}{(q, q^4, q^5; q^5)_\infty}$$

Lemma 2.9 ([1], p. 303, entry. 17(v)). *We have*

$$f_1 = f_{49} \left(\frac{B(q^7)}{C(q^7)} - q \frac{A(q^7)}{B(q^7)} - q^2 + q^5 \frac{C(q^7)}{A(q^7)} \right)$$

where

$$A(q) = f(-q^3, -q^4), B(q) = f(-q^2, -q^5) \text{ and } C(q) = f(-q, -q^6)$$

Lemma 2.10. *For all primes p and all $k, m, j \geq 1$, we have*

$$f_{pm}^{p^{k-1}j} \equiv f_m^{p^kj} \pmod{p^k} \quad (2.17)$$

3. Congruence of $\bar{b}_5(n)$

Theorem 3.1. *For any $n \geq 0$ we have*

$$\bar{b}_5(2n) \equiv \bar{b}_5(n) \pmod{4} \quad (3.1)$$

Proof. Setting $r = 5$, in (1.11), we have

$$\sum_{n=0}^{\infty} \bar{b}_5(n) q^n = \frac{f_4^5}{f_1^{10} f_2^5} = \frac{f_4^5}{f_2^5} \left(\frac{1}{f_1^8} \right) \left(\frac{1}{f_1^2} \right) \quad (3.2)$$

Using (2.10) and (2.11) in (3.2) and expanding the expression and collecting the terms that include q^{2n} and q^{2n+1} , we have

$$\sum_{n=0}^{\infty} \bar{b}_5(2n) q^n = \frac{f_2^{33}}{f_1^{38} f_4^3 f_8^2} + 16q \frac{f_2^{23} f_8^2}{f_1^{34} f_4} + 16q \frac{f_2^9 f_4^{13}}{f_1^{30} f_8^2} \quad (3.3)$$

and

$$\sum_{n=0}^{\infty} \bar{b}_5(2n+1) q^n = 2 \frac{f_2^{35} f_8^2}{f_1^{38} f_4^9} + 8 \frac{f_2^{21} f_4^5}{f_1^{34} f_8^2} + 32q \frac{f_2^{11} f_4^7 f_8^2}{f_1^{30}} \quad (3.4)$$

From (3.3) we have

$$\sum_{n=0}^{\infty} \bar{b}_5(2n) q^n \equiv \frac{f_4^4}{f_1^8 f_2^4} \cdot \frac{f_4}{f_1^2 f_2} \pmod{4} \equiv \frac{f_4}{f_1^2 f_2} \pmod{4} \quad (3.5)$$

Again, using Lemma 2.10 and (3.3), we have

$$\sum_{n=0}^{\infty} \bar{b}_5(2n)q^n \equiv \frac{f_2^{33}}{f_1^{38} f_4^3 f_8^2} \pmod{4} \equiv \frac{f_4}{f_1^2 f_2} \pmod{4} \quad (3.6)$$

from (3.5) and (3.6), we obtain (3.1).

Theorem 3.2. For any $n \geq 0$, and $i \in \{3, 5, 7\}$, we have

$$\sum_{n=0}^{\infty} \bar{b}_5(8n+1)q^n \equiv 2f_1^3 \pmod{4} \quad (3.7)$$

$$\bar{b}_5(4n+3) \equiv 0 \pmod{4} \quad (3.8)$$

$$\bar{b}_5(8n+i) \equiv 0 \pmod{4} \quad (3.9)$$

Proof. From (3.4) and thanks to Lemma 2.10, we have

$$\sum_{n=0}^{\infty} \bar{b}_5(2n+1)q^n \equiv 2 \frac{f_4^8 f_8^2}{f_4^9} \pmod{4} \equiv 2f_4^3 \pmod{4} \quad (3.10)$$

From (3.10), collecting the co-efficient of q^{4n} we obtain (3.7)

Since f_4^3 is a function of q^4 , therefore (3.8) and (3.9) will be obtained from (3.10) by extracting the terms that contains q^{2n+1} , q^{4n+1} , q^{4n+2} and q^{4n+3} respectively.

Corollary 3.3. For all integer, $n \geq 0$, $\gamma \geq 0$, and $r \in \{3, 5, 7\}$, we have

$$\bar{b}_5(2^\gamma(4n+3)) \equiv 0 \pmod{4}$$

$$\bar{b}_5(2^\gamma(8n+r)) \equiv 0 \pmod{4}$$

Proof. By using Theorem 3.1 in (3.8) and (3.9) and applying induction on γ , the results follows.

Theorem 3.4 Suppose $p \geq 3$ is any prime. Then for any integer $\alpha \geq 0$, we have

$$\sum_{n=0}^{\infty} \bar{b}_5(8.p^{2\alpha}.n + p^{2\alpha})q^n \equiv 2\psi(q) \pmod{4} \quad (3.11)$$

$$\bar{b}_5(8.p^{2\alpha+2}.n + (8i+p)p^{2\alpha+1}) \equiv 0 \pmod{4} \quad (3.12)$$

Proof. From (3.7), we have

$$\sum_{n=0}^{\infty} \bar{b}_5(8n+1)q^n \equiv 2f_1^3 \pmod{4} \equiv 2\psi(q) \pmod{4} \quad (3.13)$$

(3.13) shows that (3.11) is valid for $\alpha = 0$. Let (3.11) be valid for some fixed $\alpha \geq 0$. Using Lemma 2.5, (3.11) implies

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot p^{2\alpha} \cdot n + p^{2\alpha}) q^n \equiv 2 \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}} \psi(q^{p^2}) \pmod{4}$$

Extracting the terms involving $q^{pn+\frac{p^2-1}{8}}$, dividing both sides by $q^{\frac{p^2-1}{8}}$ and replacing q^p by q , we obtain

$$\sum_{n=0}^{\infty} \bar{b}_5\left(8 \cdot p^{2\alpha}\left(pn + \frac{p^2-1}{8}\right) + p^{2\alpha}\right) q^n \equiv 2\psi(q^p) \pmod{4}$$

which yields

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot p^{2\alpha+1} \cdot n + p^{2\alpha+2}) q^n \equiv 2\psi(q^p) \pmod{4} \quad (3.14)$$

Extracting the terms involving q^{pn} from both sides of (3.14), replacing q^p by q , we obtain

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot p^{2\alpha+2} \cdot n + p^{2\alpha+2}) q^n \equiv 2\psi(q) \pmod{4}$$

Thus (3.11) holds for $\alpha + 1$. This completes the proof of (3.11).

Similarly, extracting the terms that involve for q^{pn+i} , $1 \leq i \leq p-1$, from (3.14), we arrive at (3.12).

Theorem 3.5. If $u_1 \in \{11, 19\}$, $u_2 \in \{17, 33\}$, $u_3 \in \{13, 21, 29, 37\}$, $u_4 \in \{15, 23, 31, 39, 47, 55\}$, then for all integers $n \geq 0$ and for all $\alpha, \beta, \gamma \geq 0$, we have

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}) q^n \equiv 2f_1^3 \pmod{4} \quad (3.15)$$

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}) q^n \equiv \begin{cases} 2 \pmod{4} & \text{if } n = \frac{k(3k-1)}{2} \\ 0 \pmod{4} & \text{otherwise} \end{cases} \quad (3.16)$$

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma}) q^n \equiv 2f_3^3 \pmod{4} \quad (3.17)$$

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + u_1 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma}) q^n \equiv 0 \pmod{4} \quad (3.18)$$

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma}) q^n \equiv 2f_5^3 \pmod{4} \quad (3.19)$$

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma}) q^n \equiv 2f_1^3 \pmod{4} \quad (3.20)$$

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + u_2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}) q^n \equiv 0 \pmod{4} \quad (3.21)$$

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + u_3 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}) q^n \equiv 0 \pmod{4} \quad (3.22)$$

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} n + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}) q^n \equiv 2f_7^3 \pmod{4} \quad (3.23)$$

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} n + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}) q^n \equiv 2f_1^3 \pmod{4} \quad (3.24)$$

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} n + u_4 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}) q^n \equiv 0 \pmod{4} \quad (3.25)$$

Proof. (3.7) is the $\alpha = \beta = \gamma = 0$ case of (3.15). Let for some integer $\alpha \geq 0$ and $\beta = \gamma = 0$, (3.15) is true. Then from (3.15), we have

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot 3^{2\alpha} \cdot n + 3^{2\alpha}) q^n \equiv 2f_1^3 \pmod{4} \quad (3.26)$$

Using Lemmas 2.4 and 2.10 in (3.26), we have

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot 3^{2\alpha} \cdot n + 3^{2\alpha}) q^n \equiv 2(f_3 - 3qf_9^3) \pmod{4} \quad (3.27)$$

Extracting the terms that include q^{3n+1} from (3.27) and replacing q^3 by q , we obtain

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot 3^{2\alpha+1} \cdot n + 3^{2\alpha+2}) q^n \equiv 2f_3^3 \pmod{4} \quad (3.28)$$

Collecting the terms that includes q^{3n} from (3.28), we have

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot 3^{2\alpha+2} \cdot n + 3^{2\alpha+2}) q^n \equiv 2f_1^3 \pmod{4} \quad (3.29)$$

Thus (3.15) is true for $\alpha + 1$ when $\beta = \gamma = 0$. Consequently, (3.15) holds for all $\alpha \geq 0$ when $\beta = \gamma = 0$.

Assuming $\gamma = 0$, we suppose that (3.15) holds any integers $\alpha, \beta \geq 0$. Using Lemma 2.8 in (3.15) when $\gamma = 0$, and gathering the terms of the type q^{5n+3} and dividing by q^3 , we obtain

$$\sum_{n=0}^{\infty} \bar{b}_5 (8 \cdot 3^{2\alpha} 5^{2\beta+1} \cdot n + 3^{2\alpha} \cdot 5^{2\beta+2}) q^n \equiv 2f_5^3 \pmod{4} \quad (3.30)$$

Again, extracting the terms involving q^{5n} from both sides of (3.30), we obtain

$$\sum_{n=0}^{\infty} \bar{b}_5 (8 \cdot 3^{2\alpha} 5^{2\beta+2} \cdot n + 3^{2\alpha} \cdot 5^{2\beta+2}) q^n \equiv 2f_1^5 \pmod{4} \quad (3.31)$$

Thus, (3.15) is true for $\beta + 1$.

Finally, we assume that the congruence holds for all $\alpha, \beta \geq 0$ and some integer $\gamma \geq 0$. Utilizing Lemma.2.9 in (3.15) then collecting the terms of the type q^{7n+6} , dividing by q^6 , we have

$$\sum_{n=0}^{\infty} \bar{b}_5 (8 \cdot 3^{2\alpha} 5^{2\beta} \cdot 7^{2\gamma+1} n + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}) q^n \equiv 2f_7^3 \pmod{4} \quad (3.31)$$

Collecting the entries that contains q^{7n} from (3.31), we obtain

$$\sum_{n=0}^{\infty} \bar{b}_5 (8 \cdot 3^{2\alpha} 5^{2\beta} \cdot 7^{2\gamma+2} n + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}) q^n \equiv 2f_1^3 \pmod{4} \quad (2.32)$$

Thus, (3.23) holds for $\gamma + 1$. Therefore, (3.23) holds for all $\alpha, \beta, \gamma \geq 0$.

By applying Lemmas 2.4 and 2.10 to equation (3.15), we arrive at

$$\sum_{n=0}^{\infty} \bar{b}_5 (8 \cdot 3^{2\alpha} 5^{2\beta} \cdot 7^{2\gamma} n + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}) q^n \equiv 2(f_3 - 3qf_9^3) \pmod{4} \quad (3.33)$$

Extracting the members that include q^{3n} and replacing q^3 by q , we obtain

$$\sum_{n=0}^{\infty} \bar{b}_5 (8 \cdot 3^{2\alpha+1} 5^{2\beta} \cdot 7^{2\gamma} n + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}) q^n \equiv 2f_1 \pmod{4} \quad (3.34)$$

Now using (2.5) in (3.34), we obtain (3.16)

Collecting the entries of the type q^{3n+1} from (3.33), we obtain (3.17)

Collecting the entries of the type q^{3n+1} and q^{3n+2} from (3.17), we obtain for $i = 1, 2$

$$\sum_{n=0}^{\infty} \bar{b}_5 (8 \cdot 3^{2\alpha+1} 5^{2\beta} \cdot 7^{2\gamma} (3n+i) + 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma}) q^n \equiv 0 \pmod{4} \quad (3.35)$$

which implies (3.18)

Employing Lemma 2.8 in (3.15), we obtain

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot 3^{2\alpha} 5^{2\beta} \cdot 7^{2\gamma} n + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}) q^n \equiv 2f_{25}^3 \left(a(q^5) - q - \frac{q^2}{a(q^5)} \right)^3 \pmod{4} \quad (3.36)$$

Collecting the entries of the type q^{5n+3} from (3.36) and replacing q^5 by q , we obtain

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma}) q^n \equiv 2f_5^3 \pmod{4} \quad (3.37)$$

which implies (3.19).

Collecting the entries of the type q^{5n} from (3.37), we obtain (3.20).

Moreover, extracting the terms that contains q^{5n+2} and q^{5n+4} from (3.36), we obtain (3.21). Again, extracting the terms involving q^{5n+j} for $j = 1, 2, 3, 4$ from (3.19), we obtain (3.22)

Now employing the Lemma.2.9 in (3.15) and collecting the terms of the type q^{7n+6} , we get (3.23) We again collect the terms of the form q^{7n} from both sides of (3.23), we arrive at (3.24). Finally, we extract the terms of the form q^{7n+l} , for $l = 1, 2, 3, 4, 5, 6$, we obtain (3.25)

Theorem 3.6. For any prime p with $\left(-\frac{2}{p}\right) = -1$ and u any integer such that $1 \leq u \leq p-1$, then for all integer $n \geq 0$ and all integer $\alpha \geq 0$, we have

$$\sum_{n=0}^{\infty} \bar{b}_5(8n+1)q^n \equiv 4\psi(q)\psi(q^2) \pmod{8} \quad (3.38)$$

$$\bar{b}_5(8n+7) \equiv 0 \pmod{8} \quad (3.39)$$

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot p^{2\alpha} n + 3 \cdot p^{2\alpha}) q^n \equiv 4\psi(q)\psi(q^2) \pmod{8} \quad (3.40)$$

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot p^{2\alpha+2} n + (8u+3p) \cdot p^{2\alpha+1}) q^n \equiv 0 \pmod{8} \quad (3.41)$$

Proof. Using Lemma 2.10 in (3.4), we have

$$\sum_{n=0}^{\infty} \bar{b}_5(2n+1)q^n \equiv 2 \frac{f_2^2 f_4}{f_1^2} \pmod{8} \quad (3.42)$$

From (2.10) in (3.42) then collecting the terms of the form q^{2n+1} we have

$$\sum_{n=0}^{\infty} \bar{b}_5(4n+3)q^n \equiv 4 \frac{f_2^3 f_8^2}{f_4} \pmod{8} \quad (3.43)$$

Extracting the terms that includes q^{2n} from both sides of (3.43) and replacing q^2 by q , we obtain

$$\sum_{n=0}^{\infty} \bar{b}_5(8n+3)q^n \equiv 4 \frac{f_1^3 f_4^2}{f_2} \pmod{8} \equiv 4 \psi(q)\psi(q^2) \pmod{8} \quad (3.44)$$

Extracting the terms containing q^{2n+1} from (3.43), we arrive at (3.39)

(3.44) shows that (3.40) is valid for $\alpha = 0$. Let for some $\alpha \geq 0$, (3.40) is valid. Using Lemma 2.5 in (3.40) we have

$$\begin{aligned} \sum_{n=0}^{\infty} \bar{b}_5(8.p^{2\alpha}n + 3.p^{2\alpha})q^n &\equiv 4 \left[\sum_{x=0}^{\frac{p-3}{2}} q^{\frac{x^2+x}{2}} f\left(q^{\frac{p^2+(2x+1)p}{2}}, q^{\frac{p^2-(2x+1)p}{2}}\right) + q^{\frac{p^2-1}{8}} \psi(q^{p^2}) \right] X \\ &\quad \left[\sum_{k=0}^{\frac{p-3}{2}} q^{y^2+y} f(q^{p^2+(2y+1)p}, q^{p^2-(2y+1)p}) + q^{\frac{p^2-1}{4}} \psi(q^{2p^2}) \right] \pmod{8} \end{aligned} \quad (3.45)$$

Consider the congruence

$$\frac{x^2+x}{2} + (y^2+y) \equiv \frac{3(p^2-1)}{8} \pmod{p}$$

which is equivalent to

$$(2x+1)^2 + 2(2y+1) \equiv 0 \pmod{p} \quad (3.46)$$

Since, $\left(\frac{-2}{p}\right) = -1$, the congruence (3.45) has only the solution $x = y = \frac{p-1}{2}$.

Extract the terms of the form $q^{pn} \frac{3(p^2-1)}{8}$ from (3.45), dividing both sides by $q^{\frac{3(p^2-1)}{8}}$ and then replace q^p with q , we obtain

$$\sum_{n=0}^{\infty} \bar{b}_5(8.p^{2\alpha+1}n + 3.p^{2\alpha+2})q^n \equiv 4 \psi(q^p)\psi(q^{2p}) \pmod{8} \quad (3.47)$$

Extracting all the terms that involve q^{pn} from (3.47) and replacing q^p by q , we have

$$\sum_{n=0}^{\infty} \bar{b}_5(8.p^{2\alpha+2}n + 3.p^{2\alpha+2})q^n \equiv 4 \psi(q)\psi(q^2) \pmod{8}$$

Thus, (3.40) is true for $\alpha + 1$. This completes the proof of (3.40)

Similarly Extracting the terms that involve q^{pn+u} where $1 \leq u \leq p-1$, from (3.47), we obtain (3.41).

Theorem 3.7. For any prime p with $\left(\frac{-2}{p}\right) = -1$ and j any integer such that $1 \leq j \leq p-1$, then for all integers $n \geq 0$ and $\alpha \geq 0$, we have

$$\bar{b}_5(8n+7) \equiv 0 \pmod{16} \quad (3.48)$$

$$\sum_{n=0}^{\infty} \bar{b}_5(8n+3)q^n \equiv 12\psi(q)f_2^3 \pmod{16} \quad (3.49)$$

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot p^{2\alpha} \cdot n + 3 \cdot p^{2\alpha})q^n \equiv 12p^\alpha(-1)^\alpha \cdot (p-1)^{1/2} \psi(q)f_2^3 \pmod{16} \quad (3.50)$$

$$\sum_{n=0}^{\infty} \bar{b}_5(8 \cdot p^{2\alpha+1}(pn+j) + 3 \cdot p^{2\alpha+2})q^n \equiv 0 \pmod{16} \quad (3.51)$$

Proof. Setting $r = 5$ in (1.11) and using (2.8) we have

$$\sum_{n=0}^{\infty} \bar{b}_5(n)q^n = \frac{f_4^5}{f_2^5} \left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8} \right)^5 \quad (3.52)$$

Extracting the term of the form q^{2n+1} , dividing throughout by q , we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \bar{b}_5(n)q^n &= 10 \frac{f_2^7 f_4^{19}}{f_1^{30} f_8^6} + 80 \frac{f_2^{11} f_4^7 f_8^2}{f_1^{30}} + 32q^2 \frac{f_2^{15} f_8^{10}}{f_1^{30} f_4^5} \\ &\equiv 10 \frac{f_8^2 f_1^2}{f_2 f_4} \pmod{16} \end{aligned}$$

Using (2.8) in (3.62), collect the terms of the type q^{2n+1} , we obtain

$$\sum_{n=0}^{\infty} \bar{b}_5(4n+3)q^n = 12 \frac{f_4 f_8^2}{f_2} \pmod{16} \quad (3.53)$$

Extracting the terms that involve q^{2n+1} from (3.53) we get (3.48).

Extracting the terms containing q^{2n} from both sides of (3.53), we have

$$\sum_{n=0}^{\infty} \bar{b}_5(8n+3)q^n = 12 \frac{f_2 f_4^2}{f_1} \pmod{16} \equiv 12 \psi(q)f_2^3 \pmod{16} \quad (3.54)$$

(3.54) is the $\alpha = 0$ case of (3.50). Assume (3.50) is valid for some $\alpha \geq 0$. Using Lemma 2.5 and Lemma 2.7, we have

$$\sum_{n=0}^{\infty} \bar{b}_5(8.p^{2\alpha}.n + 3.p^{2\alpha})q^n$$

$$\equiv 12p^\alpha(-1)^{\alpha.(p-1)/2} \left[\sum_{x=0}^{\frac{p-3}{2}} q^{\frac{x^2+x}{2}} f\left(q^{\frac{p^2+(2x+1)p}{2}}, q^{\frac{p^2-(2y+1)p}{2}}\right) + q^{\frac{p^2-1}{8}} \psi(q^{p^2}) \right] X$$

$$\left[\sum_{\substack{y=0 \\ y \neq \frac{p-1}{2}}}^{p-1} (-1)^k q^{y(y+1)} \sum_{n=0}^{\infty} (-1)^n (2pn + 2y + 1) q^{pn(pn+2y+1)} + (-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{4}} f_{2p^2}^3 \right] \quad (3.55)$$

Consider the congruence

$$\frac{x^2 + x}{2} + (y^2 + y) = 3 \frac{p^2 - 1}{8}$$

which is equivalent to

$$(2x + 1)^2 + 2(2y + 1)^2 \equiv 0 \pmod{p} \quad (3.56)$$

Since $\left(\frac{-2}{p}\right) = -1$, the congruence (3.56) has the only solution $x = y = \frac{p-1}{2}$. Extracting the terms that involve $q^{pn \frac{3(p^2-1)}{8}}$ from both sides of (3.55), dividing throughout by $q^{\frac{3(p^2-1)}{8}}$ and replacing q^p by q , we obtain

$$\sum_{n=0}^{\infty} \bar{b}_5(8.p^{2\alpha+1}.n + 3.p^{2\alpha+2})q^n \equiv 12 p^{\alpha+1} (-1)^{(\alpha+1).(p-1)/2} \psi(q^p) f_{2p}^3 \pmod{16} \quad (3.57)$$

By collecting the terms that involve q^{pn} and replacing q^p by q , we obtain

$$\sum_{n=0}^{\infty} \bar{b}_5(8.p^{2\alpha+2}.n + 3.p^{2\alpha+2})q^n \equiv 12 p^{\alpha+1} (-1)^{(\alpha+1).(p-1)/2} \psi(q) f_2^3 \pmod{16}$$

This shows (3.50) is valid for $\alpha + 1$. By induction, (3.50) is always true.

Finally extract the terms of the form q^{pn+j} where $1 \leq j \leq p - 11$ from (3.57), to obtain (3.51).

4. Congruence of $\bar{b}_6(n)$

Theorem 4.1. For all $n \geq 0$ we have,

$$\bar{b}_6(4n + 2) \equiv 0 \pmod{16} \quad (4.1)$$

$$\bar{b}_6(16n + 8) \equiv 0 \pmod{16} \quad (4.2)$$

$$\bar{b}_6(32n + 12) \equiv 0 \pmod{16} \quad (4.3)$$

$$\bar{b}_6(32n + 20) \equiv 0 \pmod{16} \quad (4.4)$$

$$\bar{b}_6(32n + 28) \equiv 0 \pmod{16} \quad (4.5)$$

$$\bar{b}_6(64n + 24) \equiv 0 \pmod{16} \quad (4.6)$$

$$\bar{b}_6(64n + 40) \equiv 0 \pmod{16} \quad (4.7)$$

$$\bar{b}_6(64n + 56) \equiv 0 \pmod{16} \quad (4.8)$$

$$\bar{b}_6(128n + 16) \equiv 0 \pmod{16} \quad (4.9)$$

$$\bar{b}_6(128n + 48) \equiv 0 \pmod{16} \quad (4.10)$$

$$\bar{b}_6(128n + 70) \equiv 0 \pmod{16} \quad (4.11)$$

Proof. Setting $r = 6$ in (1.8) and employing (2.12), we obtain

$$\sum_{n=0}^{\infty} \bar{b}_6(n) q^n = \frac{f_4^6}{f_2^6} \left[\frac{f_4^{42}}{f_2^{42} f_8^{12}} + 12q \frac{f_4^{30}}{f_2^{38} f_8^4} + 48q^2 \frac{f_4^{18} f_8^4}{f_2^{34}} + 64q^3 \frac{f_4^6 f_8^{12}}{f_2^{30}} \right] \quad (4.12)$$

Extracting the terms of the form q^{2n} and q^{2n+1} from (4.12), we obtain

$$\sum_{n=0}^{\infty} \bar{b}_6(2n) q^n = \frac{f_2^{48}}{f_1^{48} f_4^{12}} + 48q \frac{f_2^{24} f_4^4}{f_1^{40}} \quad (4.13)$$

and

$$\sum_{n=0}^{\infty} \bar{b}_6(2n+1) q^n = 12 \frac{f_2^{36}}{f_1^{44} f_4^4} + 64q \frac{f_2^{12} f_4^{12}}{f_1^{36}} \quad (4.14)$$

Using Lemma 2.10 in (4.13), we have

$$\sum_{n=0}^{\infty} \bar{b}_6(2n) q^n \equiv \frac{f_2^{24}}{f_4^{12}} \pmod{16} \equiv \frac{f_2^8}{f_4^4} \pmod{16} \quad (4.15)$$

Gathering the terms of the form q^{2n} and q^{2n+1} from (4.15), we obtain

$$\sum_{n=0}^{\infty} \bar{b}_6(4n) q^n \equiv \frac{f_1^8}{f_2^4} \pmod{16} \quad (4.16)$$

$$\bar{b}_6(4n+2) \equiv 0 \pmod{16} \quad (4.17)$$

Using (2.10) in (4.16), we obtain

$$\sum_{n=0}^{\infty} \bar{b}_6(4n) q^n \equiv \frac{1}{f_2^4} \left[\frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_2 f_{16}^2}{f_8} \right]^4 \pmod{16}$$

$$\equiv \frac{f_8^{20}}{f_4^8 f_{16}^8} + 8q \frac{f_8^{14}}{f_4^6 f_{16}^4} + 8q^2 \frac{f_8^8}{f_4^4} \quad (4.18)$$

Extracting the terms of the form q^{2n} and q^{2n+1} from (4.18), we arrive at

$$\sum_{n=0}^{\infty} \bar{b}_6(8n)q^n \equiv \frac{f_4^{20}}{f_2^8 f_8^8} + 8q \frac{f_4^8}{f_2^4} \pmod{16} \quad (4.19)$$

$$\sum_{n=0}^{\infty} \bar{b}_6(8n+4)q^n \equiv 8 \frac{f_4^{14}}{f_2^6 f_8^4} \pmod{16} \quad (4.20)$$

Using Lemma 2.10 in (4.20), we have

$$\sum_{n=0}^{\infty} \bar{b}_6(8n+4)q^n \equiv 8f_4^3 \pmod{16} \quad (4.21)$$

Right hand side of (4.21) is a function of q^4 , collecting the terms that involves q^{4n+1} , q^{4n+2} , and q^{4n+3} , we obtain (4.3), (4.4) and (4.5).

From (4.19), collecting the terms of the form q^{2n} and q^{2n+1} , we have

$$\sum_{n=0}^{\infty} \bar{b}_6(16n)q^n \equiv 8 \frac{f_2^{20}}{f_1^8 f_4^8} \pmod{16} \equiv \frac{f_2^4}{f_1^8} \pmod{16} \quad (4.22)$$

$$\sum_{n=0}^{\infty} \bar{b}_6(16n+8)q^n \equiv 8f_4^3 \pmod{16} \quad (4.23)$$

Using (2.11) in (4.22), and then gathering the terms of the type q^{2n} and q^{2n+1} , we arrive at

$$\sum_{n=0}^{\infty} \bar{b}_6(32n)q^n \equiv 8 \frac{f_2^4}{f_1^8} \pmod{16} \quad (4.24)$$

$$\sum_{n=0}^{\infty} \bar{b}_6(32n+16)q^n \equiv 8f_4^3 \pmod{16} \quad (4.25)$$

Extracting the terms of the form q^{4n+1} , q^{4n+2} and q^{4n+3} from (4.25), we arrive at (4.9), (4.10) and (4.11), respectively. (4.6), (4.7) and (4.8) can be obtained on extracting the terms containing q^{4n+1} , q^{4n+2} and q^{4n+3} from (4.23)

Corollary 4.2. For all integers $n \geq 0$ and $\alpha \geq 0$, we have

$$\bar{b}_6(2^{5+\alpha}n) \equiv \bar{b}_6(16n) \pmod{16} \quad (4.26)$$

$$\bar{b}_6(2^{3+\alpha}(2n+1)) \equiv \bar{b}_6(8n+4) \pmod{16} \quad (4.27)$$

Proof. Using (4.22), (4.24) and induction on $\alpha \geq 0$, we arrive at (4.26).

Again, using (4.21), (4.23), (4.25), and induction on $\alpha \geq 0$, we arrive at (4.27)

Theorem 4.3. Suppose $p \geq 3$ is any prime. Then for all integers $n \geq 0$, $\alpha \geq 0$ and $1 \leq u \leq (p-1)$, we have

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot p^{2\alpha} \cdot n + 4 \cdot p^{2\alpha}) q^n \equiv 8 \psi(q) \pmod{16} \quad (4.28)$$

$$\bar{b}_6(32 \cdot p^{2\alpha+2} \cdot n + (4u + p) \cdot p^{2\alpha+1}) \equiv 0 \pmod{16} \quad (4.29)$$

Proof. Collecting the terms of the type q^{4n} from (4.21), we obtain

$$\sum_{n=0}^{\infty} \bar{b}_6(32n + 4) q^n \equiv 8 \psi(q) \pmod{16} \quad (4.30)$$

which shows (4.28) is true for $\alpha = 0$. We will omit the specific steps because it can be obtained using the same logic as the proof of Theorem 3.4.

Theorem 4.4. If $u_1 \in \{44, 76\}$, $u_2 \in \{68, 132\}$, $u_3 \in \{52, 84, 116, 148\}$, $u_4 \in \{60, 92, 124, 156, 188, 220\}$, then for all integer n , $\alpha, \beta, \gamma \geq 0$, we have

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 4 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}) q^n \equiv 8 f_1^3 \pmod{16} \quad (4.31)$$

$$\sum_{n=0}^{\infty} \bar{b}_6(8 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 4 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}) q^n \equiv \begin{cases} 8 \pmod{4} & \text{if } n = \frac{k(3k-1)}{2} \\ 0 \pmod{4} & \text{otherwise} \end{cases} \quad (4.32)$$

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 4 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma}) q^n \equiv 2 f_3^3 \pmod{16} \quad (4.33)$$

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + u_1 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma}) q^n \equiv 0 \pmod{16} \quad (4.34)$$

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 4 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma}) q^n \equiv 2 f_5^3 \pmod{16} \quad (4.35)$$

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + 4 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma}) q^n \equiv 2 f_1^3 \pmod{16} \quad (4.36)$$

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + u_2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}) q^n \equiv 0 \pmod{16} \quad (4.37)$$

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + u_3 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}) q^n \equiv 0 \pmod{16} \quad (4.38)$$

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} n + 4 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}) q^n \equiv 2 f_7^3 \pmod{16} \quad (4.39)$$

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} n + 4 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}) q^n \equiv 2 f_1^3 \pmod{16} \quad (4.40)$$

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} n + u_4 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1}) q^n \equiv 0 \pmod{16} \quad (4.41)$$

Proof. Collecting the terms that include q^{4n} from both sides of (4.21) then replacing q^4 by q , we have

$$\sum_{n=0}^{\infty} \bar{b}_6(32n+4)q^n \equiv 8f_1^3 \pmod{16}$$

which is the $\alpha = \beta = \gamma = 0$ case of (4.31). One can apply the same reasoning as in the proof of Theorem.3.5, to get the results in Theorem.4.4, so we will skip the detailed steps.

Theorem 4.5. Suppose $p \geq 0$ be any prime with $\left(\frac{-2}{p}\right) = -1$. For all integers $n \geq 0, \alpha \geq 0, \beta \geq 0$ and $1 \leq u \leq p-1$, we have

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha} \cdot p^{2\beta} \cdot n + 4 \cdot 3^{2\alpha} \cdot p^{2\beta})q^n \equiv 8f_1f_2 \pmod{16} \quad (4.42)$$

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha} \cdot p^{2\beta+2} \cdot n + 4(8u+p) \cdot 3^{2\alpha} \cdot p^{2\beta+1})q^n \equiv 0 \pmod{16} \quad (4.43)$$

Proof. Extracting the terms that involves q^{4n} from both sides of (4.21), and replacing q^4 by q and employing Lemma 2.10, we obtain

$$\sum_{n=0}^{\infty} \bar{b}_6(32n+4)q^n \equiv 8f_1f_2 \pmod{16} \quad (4.44)$$

which is the case $\alpha = \beta = 0$ of (4.42). Assume (4.42) holds for some positive integer $\alpha \geq 0$ and $\beta = 0$.

Employing (2.16) in (4.42) with $\beta = 0$ we obtain

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha} n + 4 \cdot 3^{2\alpha})q^n \equiv 8(f_3 - qf_9f_{18}) \pmod{16} \quad (4.45)$$

Collecting the terms that contains q^{3n+1} from both sides of (4.46),

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha+1} n + 4 \cdot 3^{2\alpha+2})q^n \equiv 8f_3f_6 \pmod{16} \quad (4.46)$$

Extracting the terms involving q^{3n} from (4.46), we obtain

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha+2} n + 4 \cdot 3^{2\alpha+2})q^n \equiv 8f_1f_2 \pmod{16} \quad (4.47)$$

This shows that for $\alpha+1$ with $\beta = 0$, the congruence (4.42) holds. We now assume that for any $\alpha \geq 0$ and for some $\beta \geq 0$, the congruence (4.42) holds

Employing Lemma.2.6 in (4.42), we obtain

$$\begin{aligned}
& \sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha+2} \cdot p^{2\beta} \cdot n + 4 \cdot 3^{2\alpha+2} \cdot p^{2\beta}) q^n \\
& \equiv 8 \left[\sum_{\substack{i=-(p-1)/2 \\ i \neq \pm(p-1)/6}}^{(p-1)/2} (-1)^i q^{(3i^2+i)/2} f(-q^{(3p^2+(6i+1)p)/2}, -q^{(3p^2-(6i+1)p)/2}) \right. \\
& \quad + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2} \left[\sum_{\substack{k=-(p-1)/2 \\ k \neq \pm(p-1)/6}}^{(p-1)/2} (-1)^k q^{(3k^2+k)} f(-q^{(3p^2+(6k+1)p}), -q^{(3p^2-(6k+1)p}) \right) \\
& \quad \left. + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{12}} f_{2p^2} \right] \pmod{16} \tag{4.48}
\end{aligned}$$

Consider the congruence

$$\left(\frac{3i^2 + i}{2} \right) + (3k^2 + k) = \frac{p^2 - 1}{8}$$

Which is equivalent to

$$(6i + 1)^2 + 2(6k + 1)^2 \equiv 0 \pmod{p}$$

For $\left(\frac{-2}{p}\right) = -1$, the above congruence has only the solution $i = k = \frac{\pm p-1}{6}$. Extracting all the terms of the form $q^{pn} \frac{p^2-1}{8}$ then dividing by $q^{\frac{p^2-1}{8}}$ and replacing q^p by q , we arrive at

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha} \cdot p^{2\beta+1} \cdot n + 4 \cdot 3^{2\alpha} \cdot p^{2\beta+2}) q^n \equiv 8 f_p f_{2p} \pmod{16} \tag{4.49}$$

Finally extracting the terms of the form q^{pn} from (4.49), we arrive at

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha} \cdot p^{2\beta+2} \cdot n + 4 \cdot 3^{2\alpha} \cdot p^{2\beta+2}) q^n \equiv 8 f_1 f_2 \pmod{16} \tag{4.50}$$

which implies for $\beta + 1$, (4.42) holds. By induction (4.42) is valid for every $\alpha \geq 0$ and $\beta \geq 0$

To prove our next congruence (4.43), we collect all the terms of the form q^{pn+u} , where $1 \leq u \leq p-1$ from both sides of (4.49).

Corollary 4.6. For all $n \geq 0$ and $\alpha \geq 0$, we have

$$\bar{b}_6(32 \cdot 3^{2\alpha} \cdot n + 68 \cdot 3^{2\alpha}) \equiv 0 \pmod{16} \tag{4.51}$$

$$\bar{b}_6(32 \cdot 3^{2\alpha+2} \cdot n + 4(8j + 3) \cdot 3^{2\alpha+1}) \equiv 0 \pmod{16} \tag{4.52}$$

Proof. By extracting the terms that include q^{3n+2} from (4.45), we get

$$(4.51)$$

Similarly, extracting only those terms that include q^{3n+j} , for $j = 1, 2$ (4.47), we obtain

$$\sum_{n=0}^{\infty} \bar{b}_6(32 \cdot 3^{2\alpha+1}(3n+j) + 4 \cdot 3^{2\alpha+2})q^n \equiv 0 \pmod{16} \quad (4.53)$$

which yields the congruence (4.52).

Theorem 4.7. Suppose $p \geq 3$ is any prime with $\left(\frac{-4}{p}\right) = -1$ and $1 \leq v \leq p-1$, then all integer $n \geq 0$ and $\alpha \geq 0$, we have

$$\bar{b}_6(4n+3) \equiv 0 \pmod{16} \quad (4.54)$$

$$\sum_{n=0}^{\infty} \bar{b}_6(8n+5)q^n \equiv 8\psi(q)\psi(q^4) \pmod{16} \quad (4.55)$$

$$\sum_{n=0}^{\infty} \bar{b}_6(8 \cdot p^{2\alpha} \cdot n + 5 \cdot p^{2\alpha})q^n \equiv 8\psi(q)\psi(q^4) \pmod{16} \quad (4.56)$$

$$\bar{b}_6(8 \cdot p^{2\alpha+1}(pn+v) + 5 \cdot p^{2\alpha+2}) \equiv 0 \pmod{16} \quad (4.57)$$

Proof. Using (4.14), and employing Lemma 2.10, we obtain

$$\sum_{n=0}^{\infty} \bar{b}_6(2n+1)q^n \equiv 12f_4^2 f_2^2 \pmod{16} \quad (4.58)$$

Extracting the terms containing q^{2n} and q^{2n+1} , replacing q^2 by q , we have

$$\sum_{n=0}^{\infty} \bar{b}_6(4n+1)q^n \equiv 12f_2^2 f_1^2 \pmod{16} \quad (4.59)$$

$$\bar{b}_6(4n+3) \equiv 0 \pmod{16}$$

Employing (2.8) in (4.59), we have

$$\sum_{n=0}^{\infty} \bar{b}_6(4n+1)q^n \equiv 12f_2^2 \left(\frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_2 f_{16}^2}{f_8} \right) \pmod{16} \quad (4.60)$$

Extract the terms of the type q^{2n+1} from (4.60) and replacing q^2 by q , we have

$$\sum_{n=0}^{\infty} \bar{b}_6(8n+5)q^n \equiv 8 \frac{f_1^3 f_8^2}{f_4} \pmod{16} \equiv 8\psi(q)\psi(q^4) \pmod{16}$$

which is the $\alpha = 0$ case (4.56). Assume (4.56) is valid for some integer $\alpha \geq 0$, then using Lemma 2.5 in (4.62), we have

$$\sum_{n=0}^{\infty} \bar{b}_6(8 \cdot p^{2\alpha} \cdot q^n + 5 \cdot p^{2\alpha}) q^n \equiv 8 \left[\sum_{x=0}^{\frac{p-3}{2}} q^{\frac{x^2+x}{2}} f\left(q^{\frac{p^2+(2x+1)p}{2}}, q^{\frac{p^2-(2x+1)p}{2}}\right) + q^{\frac{p^2-1}{8}} \psi(q^{p^2}) \right] \\ X \left[\sum_{y=0}^{\frac{p-3}{2}} q^{2(y^2+y)} f(q^{2(p^2+(2y+1)p)}, q^{2(p^2-(2y+1)p)}) + q^{\frac{p^2-1}{2}} \psi(q^{4p^2}) \right] \quad (4.61)$$

Consider the congruence

$$\frac{x^2+x}{2} + 2(y^2+y) = \frac{5(p^2-1)}{8} \quad (4.62)$$

(4.62) is equivalent to

$$(2x+1)^2 + 4(2y+1)^2 \equiv 0 \pmod{p}$$

Since $\left(\frac{-4}{p}\right) = -1$, the above congruence has only the solution $x = y = \frac{p-1}{2}$.

Extracting the terms that involve $q^{pn+\frac{5(p^2-1)}{8}}$, dividing throughout by $q^{\frac{5(p^2-1)}{8}}$ and replacing q^p by q , we have

$$\sum_{n=0}^{\infty} \bar{b}_6(8 \cdot p^{2\alpha+1} \cdot n + 5 \cdot p^{2\alpha+2}) q^n \equiv 8\psi(q^p)\psi(q^{4p}) \pmod{16} \quad (4.63)$$

Again, extracting the terms that involve q^{pn} from both sides of (4.63) and replacing q^p by q , we have

$$\sum_{n=0}^{\infty} \bar{b}_6(8 \cdot p^{2\alpha+2} \cdot n + 5 \cdot p^{2\alpha+2}) q^n \equiv 8\psi(q)\psi(q^4) \pmod{16}$$

Thus, the (4.56) holds for $\alpha + 1$. This completes the proof of (4.56).

Finally, extracting the terms that involve q^{pn+v} with $1 \leq v \leq p-1$, we obtain (4.57).

Theorem 4.8. Suppose $p \geq 5$ is any prime with $\left(\frac{-8}{p}\right) = -1$ and $1 \leq v \leq p-1$, then for all integers $n \geq 0$ and $\alpha \geq 0$, we have

$$\bar{b}_6(8n+7) \equiv 0 \pmod{32} \quad (4.64)$$

$$\sum_{n=0}^{\infty} \bar{b}_6(8n+3) q^n \equiv 16 f_1 f_8 \pmod{32} \quad (4.65)$$

$$\sum_{n=0}^{\infty} \bar{b}_6(8 \cdot p^{2\alpha} \cdot n + 3 \cdot p^{2\alpha}) q^n \equiv 16 f_1 f_8 \pmod{32} \quad (4.66)$$

$$\bar{b}_6(8 \cdot p^{2\alpha+1}(pn + v) + 3 \cdot p^{2\alpha+2}) \equiv 0 \pmod{32} \quad (4.67)$$

Proof. Using lemma 2.10 in (4.14), we have

$$\sum_{n=0}^{\infty} \bar{b}_6(2n + 1) q^n \equiv 12 f_2^4 f_1^4 \pmod{32} \quad (4.68)$$

Using (2.9) in (4.71) and collecting terms that are of the type q^{2n+1} , we have

$$\sum_{n=0}^{\infty} \bar{b}_6(4n + 3) q^n \equiv 16 f_2 f_{16} \pmod{32} \quad (4.69)$$

Extracting the terms that contains q^{2n} and q^{2n+1} from (4.69), we obtain (4.65) and (4.64) respectively.

From (4.64), (4.66) is true for $\alpha = 0$. Let (4.66) is valid for some integer $\alpha \geq 0$. Using Lemma 2.6 in (4.66), we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \bar{b}_6(8 \cdot p^{2\alpha} \cdot n + 3 \cdot p^{2\alpha}) q^n \\ & \equiv 8 \left[\sum_{\substack{x=-(p-1)/2 \\ x \neq \pm(p-1)/6}}^{(p-1)/2} (-1)^x q^{(3x^2+x)/2} f(-q^{(3p^2+(6x+1)p)/2}, -q^{(3p^2-(6x+1)p)/2}) \right. \\ & \quad + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2} \left[\sum_{\substack{y=-(p-1)/2 \\ y \neq \pm(p-1)/6}}^{(p-1)/2} (-1)^y q^{4(3y^2+y)} f(-q^{4(3p^2+(6y+1)p}), -q^{4(3p^2-(6y+1)p)}) \right. \\ & \quad \left. \left. + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{3}} f_{8p^2} \right] \right] \pmod{32} \end{aligned} \quad (4.70)$$

Consider the congruence

$$\frac{3x^2 + x}{2} + 4(3y^2 + y) \equiv \frac{3(p^2 - 1)}{8} \pmod{p} \quad (4.71)$$

Which is equivalent to

$$(6x + 1)^2 + 8(6y + 1)^2 \equiv 0 \pmod{p}$$

Since $\left(\frac{-8}{p}\right) = -1$, the above congruence has the only solution $x = y = \frac{\pm p-1}{6}$.

Extracting the terms that involve $q^{pn+\frac{3(p^2-1)}{8}}$ from both sides of (4.70) and replacing q^p by q , we arrive at

$$\sum_{n=0}^{\infty} \bar{b}_6(8.p^{2\alpha+1}.n + 3.p^{2\alpha+2})q^n \equiv 16 f_p f_{8p} \pmod{32} \quad (4.72)$$

Similarly, extracting the terms that involve q^{pn} from both sides of (4.72) and then replacing q^p by q , we obtain

$$\sum_{n=0}^{\infty} \bar{b}_6(8.p^{2\alpha+1}.n + 3.p^{2\alpha+2})q^n \equiv 16 f_1 f_8 \pmod{32}$$

Thus, (4.66) is valid for $\alpha + 1$. Therefore, by induction the (4.66) is valid for all integer $\alpha \geq 0$

Finally, extracting the terms q^{pn+v} , $1 \leq v \leq p - 1$ from both sides of (4.72), we obtain (4.67)

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