FIXED POINT THEOREM IN FUZZY METRIC SPACES FOR NON-COMPATIBLE AND WEAKLY COMPATIBLE MAPS

Mami Sharma*
Department of Mathematical Sciences, Tezpur University, Napaam, Assam.
*For correspondence. (mamisharma@yahoo.com)

Abstract: In this paper, the concept of both weakly compatible and non-compatible maps has been applied to prove common fixed point theorem in fuzzy metric spaces. A fixed point theorem for five maps has been established. These results were proved without exploiting the notion of continuity and without imposing any condition on t-norm.

Keywords: fuzzy metric space; common fixed point; non-compatible mapping; weakly compatible mapping;

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1. Introduction:

The concept of fuzzy sets was first given by Zadeh[1] in 1965. Then Kramosil and Michalek[2] introduced the concept of fuzzy metric spaces in 1975. George and Veeramani[3] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek[2]. Later several authors like Grabiec[4], G. Jungck[5], R. Vasuki[6], D. Mihet[8] obtained fixed and common fixed point theorems satisfying various contractive conditions in fuzzy metric spaces. Recently, M. Tanveer presented a note on various fixed point theorems in recent years. In the paper, the author gave a good summary of the work done on fixed point theory in fuzzy metric spaces. The fuzzy version of Banach contraction principle was first given by Grabiec [4] in 1988. This is a mile stone in developing the fixed point theory in fuzzy metric space.

In the study of fixed points of metric spaces, Pant [7, 8, 9] has initiated work using the concept of non-compatible maps in metric spaces. Recently, Dani and Sharma [10] proved common fixed point theorems for six self maps in fuzzy metric spaces using the concept of non-compatible maps. The results obtained in the fuzzy metric fixed point theory by using non-compatible maps along with weakly compatible maps are very interesting. The aim of this paper is to obtain common fixed point of mappings satisfying generalized contractive type conditions without exploiting the notion of continuity in the setting of fuzzy metric spaces.

2. Preliminaries:

Definition 2.1.[3]: A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is called a "continuous t-norm" if $([0,1],*)$ is an abelian topological monoid with unit 1 such that- $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$ for all a,b,c and dϵ [0,1]. Examples of t-norm are $a*b = ab$ and $a*b = \min\{a,b\}$

Definition 2.2. (George and Veeramani[3, 4]): The 3- tuple (X, M,*) is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions for all x,y,z ϵ X and s,t > 0 :

(FM-1) $M(x, y, 0) = 0;$
(FM-2) $M(x, y, t) = 1$ *for all* $t > 0$ *iff* $x = y;$
(FM-3) $M(x, y, t) = M(y, x, t);$
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s);$
(FM-5) $M(x, y, .): [0, \infty) \rightarrow [0,1]$ *is left continuous;*
(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$

Note that M(x, y, t) can be thought of as the degree of nearness between x and y with respect to t. We identify x = y with M(x, y, t) = 1 for all t > 0. The following example shows that every metric space induces a fuzzy metric space.

Example [3]: Let (X, d) be a metric space. Define a*b = min {a, b} and for all x, y ϵ X,

$$M(x, y, t)= \frac{t}{t+d(x,y)} \text{ for all } t > 0 \text{ with } M(x, y, 0) = 0.$$

Then (X, M,*) is a fuzzy metric space. It is called the fuzzy metric space induced by the metric space (X,d).

Lemma 2.1. (Grabiec[4]): For all x, y ϵ X, M(x, y, .) is a non-decreasing function.

Definition 2.3. (Grabiec [4]): Let (X, M, *) be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be convergent to a point x ϵ X if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all t > 0. Further, the sequence $\{x_n\}$ is said to be a Cauchy sequence in X, if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for all t > 0 and p > 0. The space is said to be complete if every Cauchy sequence in X converges to a point of X.

Remark 1: Since * is continuous, it follows from (FM-4) that the limit of a sequence in a fuzzy metric space is unique, if it exists.

Definition 2.4.: A function M is continuous in fuzzy metric space iff whenever, $\{x_n\} \to x$ and $\{y_n\} \to y$, then $\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t)$, for each t > 0.

Lemma 2.2. (George and Veeramani[3]): Let (X, M, *) be a fuzzy metric space. If there exists a number k ϵ (0, 1) such that for all x, y ϵ X and t >0, M(x,y, kt) ≥ M(x,y,t). Then, x = y.

Definition 2.5. (Pant[8]): Self mappings A and B of a fuzzy metric space (X, M, *) are said to be compatible if and only if $\lim_{n\to\infty} M(ABx_n, BAx_n, t)=1$ for all t > 0, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = p$ ϵ X.

Definition 2.5. (Pant[8]): Mappings f and g are non-compatible maps, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} f\, x_n = p = \lim_{n\to\infty} g\, x_n$ but either

$$\lim_{n\to\infty} M(f\, g\, x_n, g\, f\, x_n, t) \neq 1,$$

or the limit does not exists for all p ϵ X.

Definition 2.6 (Kramosil and Michalek[2]): Two maps A and B from a fuzzy metric space (X, M, *) into itself are said to be weakly compatible if they commute at their coincidence points, i.e. Ax = Bx implies ABx = BAx for some x ϵ X.

3. Main Results:

Theorem 3.1.: *Let A, B, C, S and T be self mappings of a complete fuzzy metric space (X, M,*). Suppose that they satisfy the following conditions:*

- ***a)*** *A (X) $\subseteq$ T(X), B(X), C(X) $\subseteq$ S(X).*
- ***b)*** *{A, S}, {B, T} are weakly compatible and {C, T} is non-compatible.*
- ***c)*** *(i) $M(Ax, By, kt) \geq \phi\left[\min\left\{ M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), \frac{M(Ax,Ty,t) + M(By,Sx,t)}{2}\right\}\right]$*

  (ii) $M(Ax, Cy, kt) \geq \min\left\{ M(Sx, Ty, t), M(Ax, Sx, t), M(Cy, Ty, t), \frac{M(Ax,Ty,t) + M(Cy,Sx,t)}{2}\right\}$

  for all x, y ϵ X, k ϵ (0,1] and ϕ ϵ Ψ. Then A, B, C, S and T have a unique fixed point in X.

Remark: Here *Ψ* is the class of all continuous implicit functions from [0,1] × [0,1] to [0,1] which are increasing in each co-ordinates. That is, for any *f ϵ Ψ, f:[0,1] × [0,1]→[0,1]* and *f(t,t)>t* for all *t*ϵ *[0,1).*

Proof: Since {C, T} is non-compatible, so there exists a sequence $\{z_n\}$ in X such that $\lim_{n\to\infty} \boldsymbol{C z_n} = \lim_{n\to\infty} \boldsymbol{T z_n} = p$ for some p ϵ X . But either $\lim_{n\to\infty} \boldsymbol{M}(CTz_n, TCz_n, t) \neq 1$ or the limit does not exists for all p ϵ X.

Since *C(X) $\subseteq$ S(X)*, corresponding to each z_n there exists x_n such that $Cz_n = Sx_n$ and $Cz_n = Sx_n \to p$ as $n\to\infty$.

We claim: $Ax_n \to p$ as $n\to\infty$.

Putting $x=x_n$, $y=z_n$ in (c) (ii), we have

$M(Ax_n, Cz_n, kt) \geq \min\{M(Sx_n, Tz_n, t), M(Ax_n, Sx_n, t), M(Cz_n, Tz_n, t), \frac{M(Axn,\ Tzn,t)+M(Czn,\ Sxn,t)}{2}\}$

Letting n→∞, we obtain

$M(\lim_{n\to\infty} Ax_n, p, kt) \geq \min\{M(p, p, t), M(\lim_{n\to\infty} Ax_n, p, t), M(p, p, t), \frac{M(\lim_{n\to\infty} Axn,p,t)+M(p,p,t)}{2}\}$

$= M(\lim_{n\to\infty} Ax_n, p, t)$

which by Lemma 2.2 implies that $\lim_{n\to\infty} Ax_n = p$.

Again, since A(X) C T(X), for each x_n there exists $y_n \in X$ such that $Ax_n = Ty_n$. Thus, from above it follows that $Ax_n = Tyn \to p$. We show $Cy_n \to p$ and $By_n \to p$.

For $x=x_n$ and $y=y_n$ the inequality (c) (i) takes the form

$M(Ax_n, By_n, kt) \geq \phi[\min\{M(Sx_n, Ty_n, t), M(Ax_n, Sx_n, t), M(By_n, Ty_n, t), \frac{M(Axn,Tyn,t)+M(Byn,Sxn,t)}{2}\}]$

Taking limit as $n\to\infty$, we obtain

$M(\lim_{n\to\infty} Ax_n,\lim_{n\to\infty} By_n,kt) \geq \phi[\min\{M(p,p,t),M(p,p,t),M(\lim_{n\to\infty} By_n,p,t),\frac{M(p,p,t)+M(\lim_{n\to\infty} Byn,p,t)}{2}\}]$

which implies that $M(p, \lim_{n\to\infty} By_n,kt) \geq M(\lim_{n\to\infty} B_n,p,t)$.

Hence by Lemma 2.2, it gives $\lim_{n\to\infty} By_n = p$.

In the similar way, using (c) (ii) we can prove that $\lim_{n\to\infty} Cy_n = p$.

Now as X is complete, there exists $u \in X$ such that $Su = p$. We show $Su = Au$.

Putting $x = u$ and $y = y_n$ (c) (i) gives:

$M(Au, By_n, kt) \geq \phi[\min\{M(Su, Ty_n, t), M(Au, Su, t), M(By_n, Ty_n, t), \frac{M(Au,Tyn,t)+M(Byn,Su,t)}{2}\}]$

Letting $n\to\infty$, we obtain

$M(Au, Su, kt) \quad \geq \phi[\min\{M(p, p, t), M(Au, Su, t), M(p, p, t), \frac{M(Au,Su,t)+M(p,Su,t)}{2}\}]$

$\geq M(Au, Su, t)$

which implies $\quad Au = Su = p$.

By the given hypothesis $A(X) \subseteq T(X)$, there exists $w \in X$ such that $A(u) = T(w) = p$.

We show $Au = Bw = Cw = p$.

Substituting $x = u$ and $y = w$ in (c) (i) , what we obtain is as follows:

$M(Au, Bw, kt) \quad \geq \phi[\min\{M(Su, Tw, t), M(Au, Su, t), M(Bw, Tw, t), \frac{M(Au,Tw,t)+M(Bw,Su,t)}{2}\}]$

i.e. $M(Au, Bw, kt) \geq \phi(M(Au, Bw, t)) \geq M(Au, Bw, t)$.

Applying Lemma 2.2 we obtain $Au = Bw = p$.

Exactly the same way, we can obtain $Cw = Au = p$

Combining these we get the following equalities:

$Au = Su = Tw = Bw = Cw = p \qquad$ (I)

As $\{A, S\}$ are weakly compatible, we have $Ap = Sp$ using (I) and definition.

Since $\{B, T\}$ is weakly compatible, we have $Bp = Tp$ using definition and (I) again.

Next, we are to show $Tp = Ap$.

For $x = p$ and $y = p$, (c) (i) takes the form

$M(Ap, Bp, kt) \quad \geq \phi[\min\{M(Sp, Tp, t), M(Ap, Sp, t), M(Bp, Tp, t), \frac{M(Ap,Tp,t)+M(Bp,Sp,t)}{2}\}]$

$= \phi[\min\{M(Ap, Bp, t), M(Ap, Sp, t), M(Bp, Tp, t), \frac{M(Ap,Bp,t)+M(Bp,Ap,t)}{2}\}]$

$= \phi(M(Ap, Bp, t)) > M(Ap, Bp, t)$

which implies $AB = Bp$.

Using (c) (ii) we can have $Ap = Cp$. Hence $Ap = Sp = Bp = Tp$ (II)

Now, it is remain to show that $Ap = p$.

Letting $x = p$ and $y = z_n$ in (c) (ii) we have

$M(Ap, Cz_n, kt) \quad \geq \min\{M(Sp, Tz_n, t), M(Ap, Sp, t), M(Cz_n, Tz_n, t), \frac{M(Ap,Tzn,t)+M(Czn,Sp,t)}{2}\}$

Taking limit as n→∞ we obtain

$M(Ap, p, kt) \quad \geq \min\{M(Sp, p, t), M(Ap, Ap, t), M(p, p, t), \frac{M(Ap,p,t)+M(p,Sp,t)}{2}\}$

$= M(Sp, p, t)$.

i.e. $Ap = p$, which together with (II) finally gives

$Ap = Sp = Bp = Tp = p$.

Thus p is the common fixed point of A, B, C, S and T in X.

The uniqueness of p can be easily verified using either of (c)(i) or (c)(ii) (as earlier procedure). □

Corollary 3.1. : Let A be a self map on a complete fuzzy metric space $(X, M, *)$ such that for some $k \in (0,1)$,

$M(Ax, Ay, kt) \geq M(x, y, t)$ *for all* $x, y \in X$, $t > 0$
Then, A has a unique fixed point in X. □

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