ON BIPOLAR FUZZY IDEALS OF SEMIRINGS WITH THRESHOLDS

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Abstract: In this paper, the fundamental concept of bipolar fuzzy ideals and their key features are investigated. The concept of bipolar fuzzy ideals with thresholds $[(\alpha^P, \alpha^N),(\beta^P, \beta^N)]$ in semirings are introduced, which extends the definition of ideals to incorporate a more flexible representation of membership degrees, and also various properties and important results of ideals of semirings are investigated in bipolar fuzzy settings.

Keywords: Bipolar fuzzy idempotent ideals; Bipolar fuzzy ideals with thresholds; Bipolar fuzzy bi-ideals with thresholds; Bipolar fuzzy quasi-ideals with thresholds

1. Introduction:

The characterization of an associative ring as both a near-ring and a semiring is extended through various algebraic concepts. These structures prove to be particularly useful in several domains of applied mathematics, including automata theory, optimization, and formal language theory. A semiring is an algebraic structure having the operations addition and multiplication in binary associative pairs, where addition is commutative and multiplication is distributive over addition. Vandiver introduced the official definition of a semiring in 1935. Since then, other researchers have intensively investigated the semiring theory to characterize its various kinds. Some significant generalizations of ring theory to the theory of semiring include fully idempotent semiring and weakly regular semiring among others. Semirings offer the most logical common generalization of the theory of rings from an algebraic perspective. The building of rational numbers using the semifield of non-negative rationals allows for more broad research in semiring theory, and the semiring of natural numbers, including the zero or not, is likely one of the oldest semiring structures ever employed by humans. It is quite natural to investigate fuzzy set theory in the context of semirings since the unit interval itself admits the structure of a semiring under the binary operations max and standard multiplication of numbers or max and min with proper meanings. Ideals are important in the study of ring theory, just as they are in the case of researching semiring theory. Iseki first suggested the ideal of a semiring in the year 1956. Later, he expanded on this idea by introducing the quasi-ideal of a semiring and proving several associated conclusions. After that, numerous more scholars began looking into the various sorts of semiring ideals, like Sen et al. who researched the p-ideals of semirings and the prime ideal of a semiring was investigated by Gupta and Chaudhari. Semirings' Q-ideal, maximal ideal, and main ideal were defined by Chaudhari and Ingale. Another new branch of mathematics that combines concepts from algebra, order theory, and discrete mathematics is tropical algebra. This algebra is an effective tool because it enables linear analysis of naturally non-linear problems. A typical non-linear system is first converted into a tropical linear system according to the general rule of this algebra, and then information about the original system is provided using tropical algebra's techniques. One of the significant semirings, the tropical semiring was first introduced by Simon, in 1978.

Theoretical computer science includes automata theory and investigations of the theoretical model of a computer hardware or software system frequently using discrete mathematics. In simple words, the theory of automata deals with the logic of computation about machines. It has been discovered that semiring structures play a crucial role in characterizing various aspects of automata and formal language theories, so it is anticipated that further research combining automata and semiring in a fuzzy setting will be beneficial in terms of its applicability in theoretical computer science.

Human decision making is founded on bipolar judgemental thinking on both a positive and a negative side. Examples of two-sided decisions include effect and side effects, collaboration and competitiveness. Consequently, a form of fuzzy set that addresses bipolar judgement was required. The bipolar fuzzy set was introduced by Zhang, in 1998. It deals with the degree of membership of an element satisfying the property and the degree of membership of that element that satisfies the counter property. Additionally, the range of the membership degree of a bipolar fuzzy set is expanded from the interval [0,1] to [-1,1]. Later, the concept of bipolar fuzzy subsemirings and ideals of semiring was introduced by Zhou and Li.

In this paper, we utilize the concept of bipolar fuzzy sets within the context of semirings. We have introduced the definitions of bipolar fuzzy ideals with thresholds, bipolar fuzzy bi-ideals with thresholds, and bipolar fuzzy quasi-ideals with thresholds, and we explore various properties associated with these ideals.

2. Basic definitions and notations:

A non-empty set R is called a semiring when two binary operations are defined, namely, addition(+) and multiplication(.) so that (R,+) is a commutative semigroup and (R,.) is a monoid with unity 1, where these two algebraic structures are connected by the following distributive laws: a(b+c) = ab + ac and (a+b)c = ac + bc .A semiring R is said to be a regular semiring if for every x ∈ R there exists a ∈ R such that x = xax. A semiring R is said to be a right weakly regular if for every x ∈ R , x ∈ $(xR)^2$. A semiring R with multiplicative identity 1 is said to be an intra-regular, if x = $\sum_{i=1}^{n} a_i\, x^2 b_i$ for any x ∈ R, where x_i, y_i ∈ R . If (S, $+_s$, $._s$) is a semiring then the subset S of R is called a subsemiring. In a non-empty subset I of a semiring R, if (I,+) is a subsemigroup of (R,+) and ra ∈ I(ar ∈ I) for all a ∈ I, r ∈ R then, it is called a left(right) ideal of R . If a subset I of R is both the left and right ideal of R, it is referred to be an ideal . A suitable ideal P of a semiring R is said to be a prime ideal of R if AB ⊆ P implies either A ⊆ P or B ⊆ P for any two ideals A and B of R.If for any x ∈ R and y ∈ I, x+y ∈ I which implies x ∈ I, an ideal of a semiring R then it is called a k-ideal . The term "quasi-ideal" of a semiring R refers to a non-empty subset I of R such that (I,+) is as subsemigroup of (R,+) which satisfies the property IR ∩ RI ⊆ R .A subsemring I of R is said to be a bi-ideal of R, if it satisfies the property IRI ⊆ R .An idempotent ideal is an ideal I of a semiring R if I^2 = I.A semiring R is termed as a fully idempotent semiring if each ideal of R is idempotent .A mapping from X to [0,1] is called a fuzzy set μ of the set X . A fuzzy point x_t of a set X is a fuzzy subset if it is of the form $x_t = \begin{cases} 1,\text{if } x=y \\ 0,\text{if } x \neq y. \end{cases}$ If $\mu(x)$ ≥t, then the fuzzy point x_t belongs to a fuzzy subset. A non-empty set X is said to be a bipolar fuzzy set if A = {(x,$\mu_A^P(x)$,$\mu_A^N(x)$):x ∈ X } where μ_A^P :X → [0,1] and μ_A^N : X → [-1,0]. Here, μ_A^P is called the positive membership degree, which gives the degree of satisfaction of x to the corresponding property of A and μ_A^N is called the negative membership degree, which gives the degree of satisfaction of x to the corresponding implicit counter property of A.

Definition 1. A bipolar fuzzy subset A=(μ_A^P, μ_A^N) of a semiring R is said to be a bipolar fuzzy subsemiring of R if

1. $\mu_A^P(x+y) \geq \mu_A^P(x) \wedge \mu_A^P(y)$
2. $\mu_A^P(xy) \geq \mu_A^P(x) \wedge \mu_A^P(y)$
3. $\mu_A^N(x+y) \leq \mu_A^N(x) \vee \mu_A^N(y)$
4. $\mu_A^N(xy) \leq \mu_A^N(x) \vee \mu_A^N(y)$ for all x, y ∈ R.

Definition 2. A bipolar fuzzy subset A=(μ_A^P, μ_A^N) of a semiring R is said to be a bipolar fuzzy left(right) ideal of R if

1. $\mu_A^P(x+y) \geq \mu_A^P(x) \wedge \mu_A^P(y)$
2. $\mu_A^P(xy) \geq \mu_A^P(y) (\mu_A^P(xy) \geq \mu_A^P(x))$
3. $\mu_A^N(x+y) \leq \mu_A^N(x) \vee \mu_A^N(y)$
4. $\mu_A^N(xy) \leq \mu_A^N(y) (\mu_A^N(xy) \leq \mu_A^N(x))$ for all x, y ∈ R.

Definition 3. Let A=(μ_A^P, μ_A^N) and B=(μ_B^P, μ_B^N) be two bipolar fuzzy subsets of a semiring R, then the sum of bipolar fuzzy subsets A and B is defined as A+B=($\mu_A^P + \mu_B^P, \mu_A^N + \mu_B^N$) for all x ∈ R, where

$$(\mu_A^P + \mu_B^P)(x)=\begin{cases} \bigvee_{x=a+b} \{\mu_A^P(a) \wedge \mu_B^P(b)\} \\ 0,\text{if x is not expressible as } x = a+b \end{cases}$$

and $$(\mu_A^N + \mu_B^N)(x)=\begin{cases} \bigwedge_{x=a+b} \{\mu_A^N(a) \vee \mu_B^N(b)\} \\ 0,\text{if x is not expressible as } x=a+b. \end{cases}$$

Definition 4. A bipolar fuzzy subset A=(μ_A^P,μ_A^N) of a semiring R is said to be a bipolar fuzzy subsemiring with thresholds[(α^P, α^N),(β^P,β^N)] of R, if for all x,y ∈ R it satisfies the following conditions:

(i) max{μ_A^P(x+y),α^P} ≥ min{μ_A^P(x),μ_A^P(y),β^P}
(ii) max{μ_A^P(xy),α^P} ≥ min{μ_A^P(x),μ_A^P(y),β^P}
(iii) min{μ_A^N(x+y),α^N} ≤ max{μ_A^N(x),μ_A^N(y),β^N}
(iv) min{μ_A^N(xy),α^N} ≤ max{μ_A^N(x),μ_A^N(y),β^N} for all x, y ∈ R.

Definition 5. A bipolar fuzzy subset A=(μ_A^P,μ_A^N) of a semiring R is said to be a bipolar fuzzy left(right) ideal with thresholds[(α^P,α^N), (β^P,β^N)] of R, if for all x,y ∈ R it satisfies the following conditions:

(i) max{μ_A^P(x+y),α^P} ≥ min{μ_A^P(x),μ_A^P(y),β^P}
(ii) min{μ_A^N(x+y),α^N} ≤ max{μ_A^N(x),μ_A^N(y),β^N}
(iii) max{μ_A^P(xy),α^P} ≥ min{μ_A^P(y),β^P} (max{μ_A^P(xy),α^P} ≥ min{μ_A^P(x),β^P})
(iv) min{μ_A^N(xy),α^N} ≤ max{μ_A^N(y),β^N} (min{μ_A^N(xy),α^N} ≤ max{μ_A^N(x),β^N})for all x, y ∈ R.

Definition 6. A bipolar fuzzy subset A=(μ_A^P,μ_A^N) of a semiring R is said to be a bipolar fuzzy bi-ideal with thresholds[(α^P,α^N),(β^P,β^N)] of R, if for all x, y ∈R it satisfies the following conditions:
(i) max{μ_A^P(x+y),α^P} ≥ min{μ_A^P(x),μ_A^P(y),β^P}
(ii) max{μ_A^P(xy),α^P} ≥ min{μ_A^P(x),μ_A^P(y),β^P}
(iii) min{μ_A^N(x+y),α^N} ≤ max{μ_A^N(x),μ_A^N(y),β^N}
(iv) min{μ_A^N(xy),α^N} ≤ max{μ_A^N(x),μ_A^N(y),β^N}
(v) max{μ_A^P(xyz),α^P} ≥ min{μ_A^P(x),μ_A^P(z),β^P}
(vi) min{μ_A^N(xyz),α^N} ≤ max{μ_A^N(x),μ_A^N(z),β^N}for all x, y ∈ R.

Definition 7. A bipolar fuzzy subset A=(μ_A^P,μ_A^N) of a semiring R is said to be a bipolar fuzzy quasi-ideal with thresholds[(α^P, α^N),(β^P,β^N)] of R, if for all x,y ∈ R it satisfies the following conditions:
(i) max{μ_A^P(x+y),α^P} ≥ min{μ_A^P(x),μ_A^P(y),β^P}
(ii) min{μ_A^N(x+y),α^N} ≤ max{μ_A^N(x),μ_A^N(y),β^N}
(iii) max{μ_A^P(x),α^P} ≥ min{$\mu_A^P \circ \mathcal{R}^P(x)$, $\mathcal{R}^P \circ \mu_A^P$(x),$\beta^P$}
(iv) min{μ_A^N(x),α^N} ≤ max{$\mu_A^N \circ \mathcal{R}^N$(x),$\mathcal{R}^N \circ \mu_A^N$(x),$\beta^N$}for all x, y ∈ R.Where $\mathcal{R} = (\mathcal{R}^P, \mathcal{R}^N)$ is the bipolar fuzzy subset of R mapping every element of R onto 1.

3.Main Result

Theorem 1. If A=(μ_A^P, μ_A^N) is a bipolar fuzzy left(right) ideal with thresholds[(α^P, α^N),(β^P, β^N)] of R, then $\mu_A^P \wedge \beta^P$ and $\mu_A^N \vee \beta^N$ is a left(right) ideal thresholds[(α^P, α^N),(β^P, β^N)] of R.

Proof. Suppose A=(μ_A^P, μ_A^N) is a bipolar fuzzy left(right) ideal with thresholds[(α^P, α^N),(β^P, β^N)] of R, then
max{$(\mu_A^P \wedge \beta^P)(x+y), \alpha^P$}

$= [\mu_A^P(x+y) \wedge \beta^P] \vee \alpha^P$

$= [\mu_A^P(x+y) \vee \alpha^P] \wedge \beta^P$

$\geq$ min{$\mu_A^P(x), \mu_A^P(y), \beta^P$}

= min{$(\mu_A^P \wedge \beta^P)(x), (\mu_A^P \wedge \beta^P)(y), \beta^P$}
and min{$(\mu_A^N \vee \beta^P)(x+y), \alpha^N$}

$= [\mu_A^N(x+y) \vee \beta^N] \wedge \alpha^N = [\mu_A^N(x+y) \wedge \alpha^N] \vee \beta^N$

$\leq max\{\mu_A^N(x), \mu_A^N(y), \beta^N\}$ = max{$(\mu_A^N \vee \beta^N)(x), (\mu_A^N \vee \beta^N)(y), \beta^N$}.

Now,

max{$(\mu_A^P \wedge \beta^P)(xy), \alpha^P$} $= [\mu_A^P(xy) \wedge \beta^P] \vee \alpha^P = [\mu_A^P(xy) \vee \alpha^P] \wedge \beta^P \geq$ min{$\mu_A^P(y), \beta^P$}
and min{$(\mu_A^N \vee \beta^N)(xy), \alpha^N$} $= [\mu_A^N(xy) \vee \beta^N] \wedge \alpha^N = [\mu_A^N(xy) \wedge \alpha^N] \vee \beta^N \leq$ max{$\mu_A^N(y), \beta^N$}.

Hence, $\mu_A^P \wedge \beta^P$ and $\mu_A^N \vee \beta^N$ is a left(right) ideal thresholds [(α^P, α^N), (β^P, β^N)] of R. □

Theorem 2. If A=(μ_A^P, μ_A^N) is a bipolar fuzzy quasi ideal with thresholds[(α^P, α^N), (β^P, β^N)] of R, then $\mu_A^P \wedge \beta^P$ and $\mu_A^N \vee \beta^N$ is a quasi ideal thresholds[(α^P, α^N), (β^P, β^N)] of R.

Theorem 3. If A=(μ_A^P, μ_A^N) is a bipolar fuzzy subset of a semiring R, which satisfies the conditions:
max{$\mu_A^P(x+y), \alpha^P$} ≥ min{$\mu_A^P(x), \mu_A^P(y), \beta^P$} and
min{$\mu_A^N(x+y), \alpha^N$} ≤ max{$\mu_A^N(x), \mu_A^N(y), \beta^N$} if and only if the following conditions are satisfied:

(i) $\mu_A^P +_{\alpha^P}^{\beta^P} \mu_A^P \leq (\mu_A^P \wedge \beta^P) \vee \alpha^P$

(ii) $\mu_A^N +_{\alpha^N}^{\beta^N} \mu_A^N \geq (\mu_A^P \vee \beta^P) \wedge \alpha^P$.

Proof. Let A=(μ_A^P, μ_A^N) satisfies the conditions: $\max\{\mu_A^P(x+y), \alpha^P\} \geq \min\{\mu_A^P(x), \mu_A^P(y), \beta^P\}$ and $\min\{\mu_A^N(x+y), \alpha^N\} \leq \max\{\mu_A^N(x), \mu_A^N(y), \beta^N\}$ and let x∈R, then

$$(\mu_A^P +_{\alpha^P}^{\beta^P} \mu_A^P)(x) = \left\{\left(\sup_{x=a+b}\{\mu_A^P(a) \wedge \mu_A^P(b)\}\right) \wedge \beta^P\right\} \vee \alpha^P$$

$$= \{(\sup_{x=a+b}\{\min(\mu_A^P(a), \mu_A^P(b), \beta^P\}) \wedge \beta^P\} \vee \alpha^P$$

$$\leq \{\left(\sup_{x=a+b}(\mu_A^P(a+b) \vee \alpha^P) \wedge \beta^P\right\} \vee \alpha^P \leq (\mu_A^P(x) \wedge \beta^P) \vee \alpha^P$$

and $(\mu_A^N +_{\alpha^N}^{\beta^N} \mu_A^N)(x)$

$$= \left\{\left(\inf_{x=a+b}\{\mu_A^N(a) \vee \mu_A^N(b)\}\right) \vee \beta^N\right\} \wedge \alpha^N$$

$$= \{(\inf_{x=a+b}\{\min(\mu_A^N(a), \mu_A^N(b), \beta^N\}) \vee \beta^N\} \wedge \alpha^N$$

$$\geq \{\left(\inf_{x=a+b}(\mu_A^N(a+b) \wedge \alpha^N) \vee \beta^N\right\} \wedge \alpha^N \geq (\mu_A^N(x) \vee \beta^N) \wedge \alpha^N.$$

Conversely, let $\mu_A^P +_{\alpha^P}^{\beta^P} \mu_A^P \leq (\mu_A^P \wedge \beta^P) \vee \alpha^P$ and $\mu_A^N +_{\alpha^N}^{\beta^N} \mu_A^N \geq (\mu_A^N \vee \beta^N) \wedge \alpha^N$ and let x,y ∈ R, then

$$\mu_A^P(x+y) \vee \alpha^P \geq (\mu_A^P(x+y) \wedge \beta^P) \vee \alpha^P \geq \left((\mu_A^P \wedge \beta^P) \vee \alpha^P\right)(x+y) \geq \left(\mu_A^P +_{\alpha^P}^{\beta^P} \mu_A^P\right)(x+y)$$

$$= \left\{\left(\sup_{x+y=a+b}\{\mu_A^P(a) \wedge \mu_A^P(b)\}\right) \wedge \beta^P\right\} \vee \alpha^P \geq (\{\mu_A^P(x) \wedge \mu_A^P(y)\} \wedge \beta^P) \vee \alpha^P \geq (\mu_A^P(x) \wedge \mu_A^P(y) \wedge \beta^P)$$

and $\mu_A^N(x+y) \wedge \alpha^N \leq (\mu_A^N(x+y) \vee \beta^N) \wedge \alpha^N \leq \left((\mu_A^N \vee \beta^N) \wedge \alpha^N\right)(x+y) \leq \left(\mu_A^N +_{\alpha^N}^{\beta^N} \mu_A^N\right)(x+y)$

$$= \left\{\left(\inf_{x+y=a+b}\{\mu_A^N(a) \vee \mu_A^N(b)\}\right) \vee \beta^N\right\} \wedge \alpha^N \leq (\{\mu_A^N(x) \vee \mu_A^N(y)\} \vee \beta^N) \wedge \alpha^N \leq (\mu_A^N(x) \vee \mu_A^N(y) \vee \beta^N).$$

Hence, A=(μ_A^P,μ_A^N) satisfies the conditions given conditions. □

Theorem 4. Let A=(μ_A^P,μ_A^N) and B=(μ_B^P,μ_B^N) are bipolar fuzzy right and left ideals with thresholds[(α^P,α^N),(β^P,β^N) of R, respectively, then the conditions $\mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_B^P \leq \mu_A^P \wedge_{\alpha^P}^{\beta^P} \mu_B^P$ and $\mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_B^N \geq \mu_A^N \vee_{\alpha^N}^{\beta^N} \mu_B^N$ are satisfied.

Proof. Suppose A=(μ_A^P,μ_A^N) and B=(μ_B^P,μ_B^N) are bipolar fuzzy right and left ideals with thresholds[(α^P,α^N),(β^P,β^N)] of R, repectively, then

$(\mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_B^P)$(x)

$=(\sup_{x=\sum_{i=1}^{n} a_i b_i} \bigwedge_{i=1}^{n} \{\mu_A^P(a_i)\wedge \mu_B^P(b_i)\})\wedge \beta^P \vee \alpha^P$

$=(\sup_{x=\sum_{i=1}^{n} a_i b_i} \bigwedge_{i=1}^{n} \{(\mu_A^P(a_i)\wedge \beta^P)\wedge(\mu_B^P(b_i)\wedge \beta^P)\})\wedge \beta^P \vee \alpha^P$

$\leq(\sup_{x=\sum_{i=1}^{n} a_i b_i} \bigwedge_{i=1}^{n} \{(\mu_A^P(a_i\ b_i)\vee \alpha^P)\wedge(\mu_B^P(a_i\ b_i)\vee \alpha^P)\})\wedge \beta^P \vee \alpha^P$

$=(\sup_{x=\sum_{i=1}^{n} a_i b_i} \bigwedge_{i=1}^{n} \{\mu_A^P(a_i\ b_i)\wedge \mu_B^P(a_i\ b_i)\})\wedge \beta^P \vee \alpha^P$

$=(\sup_{x=\sum_{i=1}^{n} a_i b_i} (\bigwedge_{i=1}^{n} \mu_A^P(a_i\ b_i))\wedge(\bigwedge_{i=1}^{n} \mu_B^P(a_i\ b_i)))\wedge \beta^P \vee \alpha^P$

$=(\sup_{x=\sum_{i=1}^{n} a_i b_i} (\bigwedge_{i=1}^{n} \mu_A^P(a_i\ b_i)\wedge \beta^P)\wedge(\bigwedge_{i=1}^{n} \mu_B^P(a_i\ b_i)\wedge \beta^P))\wedge \beta^P \vee \alpha^P$

$\leq(\sup_{x=\sum_{i=1}^{n} a_i b_i} (\mu_A^P(\sum_{i=1}^{n} a_i\ b_i)\vee \alpha^P)\wedge(\mu_B^P(\sum_{i=1}^{n} a_i\ b_i)\vee \alpha^P))\wedge \beta^P \vee \alpha^P$

$\leq(\sup_{x=\sum_{i=1}^{n} a_i b_i} \mu_A^P(\sum_{i=1}^{n} a_i\ b_i)\wedge \mu_B^P(\sum_{i=1}^{n} a_i\ b_i))\wedge \beta^P \vee \alpha^P \leq \{\mu_A^P(\mathrm{x})\wedge \mu_B^P(\mathrm{x})\wedge \beta^P\} \vee \alpha^P$

and $(\mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_B^N)(\mathrm{x})$

$=(\inf_{x=\sum_{i=1}^{n} a_i b_i} \bigvee_{i=1}^{n} \{\mu_A^N(a_i)\vee \mu_B^N(b_i)\})\vee \beta^N \wedge \alpha^N$

$=(\inf_{x=\sum_{i=1}^{n} a_i b_i} \bigvee_{i=1}^{n} \{(\mu_A^N(a_i)\vee \beta^N)\vee(\mu_B^N(b_i)\vee \beta^N)\})\vee \beta^N \wedge \alpha^N$

$\geq(\inf_{x=\sum_{i=1}^{n} a_i b_i} \bigvee_{i=1}^{n} \{(\mu_A^N(a_i\ b_i)\wedge \alpha^N)\vee(\mu_B^N(a_i\ b_i)\wedge \alpha^N)\})\vee \beta^N \wedge \alpha^N$

$=(\inf_{x=\sum_{i=1}^{n} a_i b_i} \bigvee_{i=1}^{n} \{\mu_A^N(a_i\ b_i)\vee \mu_B^N(a_i\ b_i)\})\vee \beta^N \wedge \alpha^N$

$=(\inf_{x=\sum_{i=1}^{n} a_i b_i} (\bigvee_{i=1}^{n} \mu_A^N(a_i\ b_i))\vee(\bigvee_{i=1}^{n} \mu_B^N(a_i\ b_i)))\vee \beta^N \wedge \alpha^N$

$=(\inf_{x=\sum_{i=1}^{n} a_i b_i} (\bigvee_{i=1}^{n} \mu_A^N(a_i\ b_i)\vee \beta^N)\vee(\bigvee_{i=1}^{n} \mu_B^N(a_i\ b_i)\vee \beta^N))\vee \beta^N \wedge \alpha^N$

$\geq(\inf_{x=\sum_{i=1}^{n} a_i b_i} (\mu_A^N(\sum_{i=1}^{n} a_i\ b_i)\wedge \alpha^N)\vee(\mu_B^N(\sum_{i=1}^{n} a_i\ b_i)\wedge \alpha^N))\vee \beta^N \wedge \alpha^N$

$\geq(\inf_{x=\sum_{i=1}^{n} a_i b_i} \mu_A^N(\sum_{i=1}^{n} a_i\ b_i)\vee \mu_B^N(\sum_{i=1}^{n} a_i\ b_i))\vee \beta^N \wedge \alpha^N \geq \{\mu_A^N(\mathrm{x})\vee \mu_B^N(\mathrm{x})\vee \beta^N\} \wedge \alpha^N.$

Hence, $\mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_B^P \leq \mu_A^P \wedge_{\alpha^P}^{\beta^P} \mu_B^P$ and $\mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_B^N \geq \mu_A^N \vee_{\alpha^N}^{\beta^N} \mu_B^N$. □

Theorem 5. For a regular semiring R, if A=(μ_A^P,μ_B^N) and B= (μ_B^P,μ_B^N) are bipolar fuzzy right and left ideals with thresholds[(α^P, α^N),(β^P,β^N)] of R, respectively, then $\mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_B^P$=$\mu_A^P \wedge_{\alpha^P}^{\beta^P} \mu_B^P$ and $\mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_B^N$=$\mu_A^N \vee_{\alpha^N}^{\beta^N} \mu_B^N$.

Proof. Suppose A=(μ_A^P,μ_B^N) and B=(μ_B^P,μ_B^N) are bipolar fuzzy right and left ideals with thresholds[(α^P,α^N),(β^P,β^N)] of R respectively. Then from Theorem 4.1.5, we have $\mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_B^P \leq \mu_A^P \wedge_{\alpha^P}^{\beta^P} \mu_B^P$ and $\mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_B^N \geq \mu_A^N \vee_{\alpha^N}^{\beta^N} \mu_B^N$. Let a∈R. Since R is regular, there exists x∈R such that a = axa. Thus,

($\mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_B^P$)(a)

$=(\sup_{x=\sum_{i=1}^{n} a_i b_i} \bigwedge_{i=1}^{n} \mu_A^P(a_i) \wedge \mu_B^P(b_i)) \wedge \beta^P \vee \alpha^P$

$\geq \mu_A^P$(a)$\wedge$ μ_B^P(xa)$\wedge$ $\beta^P \vee \alpha^P$

=μ_A^P(a)$\wedge$(μ_B^P(xa)$\vee$ α^P)$\wedge$ $\beta^P \vee \alpha^P$

$\geq \mu_A^P$(a)$\wedge$(μ_B^P(a)$\wedge$ β^P)$\wedge$ $\beta^P \vee \alpha^P$

=μ_A^P(a)$\wedge$ μ_B^P(a)$\wedge$ $\beta^P \vee \alpha^P$

=($\mu_A^P \wedge_{\alpha^P}^{\beta^P} \mu_B^P$)(a)

and ($\mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_B^N$)(a)

$=(\inf_{x=\sum_{i=1}^{n} a_i b_i} \bigvee_{i=1}^{n} \mu_A^N(a_i) \vee \mu_B^N(b_i)) \vee \beta^N \wedge \alpha^N$

$\leq \mu_A^N$(a)$\vee$ μ_B^N(xa)$\vee$ $\beta^N \wedge \alpha^N$

=μ_A^N(a)$\vee$(μ_B^N(xa)$\wedge$ α^N)$\vee$ $\beta^N \wedge \alpha^N$

$\leq \mu_A^N$(a)$\vee$(μ_B^N(a)$\vee$ β^N)$\vee$ $\beta^N \wedge \alpha^N$

=μ_A^N(a)$\vee$ μ_B^N(a)$\vee$ $\beta^N \wedge \alpha^N$=($\mu_A^N \vee_{\alpha^N}^{\beta^N} \mu_B^N$)(a).

Therefore, ($\mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_B^P$)=($\mu_A^P \wedge_{\alpha^P}^{\beta^P} \mu_B^P$) and ($\mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_B^N$)=($\mu_A^N \vee_{\alpha^N}^{\beta^N} \mu_B^N$). □

Theorem 6. Any bipolar fuzzy bi-ideal A=(μ_A^P,μ_A^N) and any bipolar fuzzy ideal B=(μ_B^P,μ_B^N) with thresholds[(α^P,α^N),(β^P,β^N)] of a regular semiring R, then it satisfies the following conditions:
(i) ($\mu_A^P \wedge_{\alpha^P}^{\beta^P} \mu_B^P$) ≤ ($\mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_B^P \circ_{\alpha^P}^{\beta^P} \mu_A^P$)
(ii) ($\mu_A^N \vee_{\alpha^N}^{\beta^N} \mu_B^N$) ≥ ($\mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_B^N \circ_{\alpha^N}^{\beta^N} \mu_A^N$).

Proof. Let a bipolar fuzzy bi-ideal be A=(μ_A^P,μ_A^N) and a bipolar fuzzy ideal be B=(μ_B^P,μ_B^N) with thresholds[(α^P,α^N),(β^P,β^N)] of a semiring R and x ∈ R. Since R is regular, there exists a ∈ R such that x = xax. Now,

($\mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_B^P \circ_{\alpha^P}^{\beta^P} \mu_A^P$)(x)

$=(\sup_{x=\sum_{i=1}^{n} a_i b_i} \bigwedge_{i=1}^{n} (\mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_B^P)(a_i) \wedge \mu_A^P(b_i)) \wedge \beta^P \vee \alpha^P$

$\geq (\mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_B^P)(\mathrm{xa}) \wedge \mu_A^P(\mathrm{x}) \wedge \beta^P \vee \alpha^P$

$=((\sup_{xa=\sum_{j=1}^{m} c_j d_j} \bigwedge_{j=1}^{m} (\mu_A^P(c_j) \wedge \mu_B^P(d_j)) \wedge \beta^P \vee \alpha^P) \wedge \mu_A^P(\mathrm{a}) \wedge \beta^P \vee \alpha^P$

$\geq \mu_A^P(\mathrm{x}) \wedge \mu_B^P(\mathrm{axa}) \wedge \beta^P \vee \alpha^P \wedge \mu_A^P(\mathrm{a}) \wedge \beta^P \vee \alpha^P$

$\geq \mu_A^P(\mathrm{x}) \wedge \mu_B^P(\mathrm{x}) \wedge \beta^P \vee \alpha^P = (\mu_A^P \wedge_{\alpha^P}^{\beta^P} \mu_B^P)(\mathrm{x})$

and $(\mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_B^N \circ_{\alpha^N}^{\beta^N} \mu_A^N)(\mathrm{x})$

$=(\inf_{x=\sum_{i=1}^{n} a_i b_i} \bigvee_{i=1}^{n} (\mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_B^N)(a_i) \vee \mu_A^N(b_i)) \vee \beta^N \wedge \alpha^N$

$\leq (\mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_B^N)(\mathrm{xa}) \vee \mu_A^N(\mathrm{x}) \vee \beta^N \wedge \alpha^N$

$=((\inf_{xa=\sum_{j=1}^{m} c_j d_j} \bigvee_{j=1}^{m} (\mu_A^N(c_j) \vee \mu_B^N(d_j)) \vee \beta^N \wedge \alpha^N) \vee \mu_A^N(\mathrm{a}) \vee \beta^N \wedge \alpha^N$

$\leq \mu_A^N(\mathrm{x}) \vee \mu_B^N(\mathrm{axa}) \vee \beta^N \wedge \alpha^N \vee \mu_A^N(\mathrm{a}) \vee \beta^N \wedge \alpha^N$

$\leq \mu_A^N(\mathrm{x}) \vee \mu_B^N(\mathrm{x}) \vee \beta^N \wedge \alpha^N = (\mu_A^N \vee_{\alpha^N}^{\beta^N} \mu_B^N)(\mathrm{x})$.

Hence, $(\mu_A^P \wedge_{\alpha^P}^{\beta^P} \mu_B^P) \leq (\mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_B^P \circ_{\alpha^P}^{\beta^P} \mu_A^P)$ and $(\mu_A^N \vee_{\alpha^N}^{\beta^N} \mu_B^N) \geq (\mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_B^N \circ_{\alpha^N}^{\beta^N} \mu_A^N)$. □

Theorem 7. Any bipolar fuzzy bi-ideal A=(μ_A^P,μ_A^N) and any bipolar fuzzy left ideal B=(μ_B^P,μ_B^N) with thresholds[(α^P,α^N),(β^P,β^N)] of a regular semiring R, then it satisfies $(\mu_A^P \wedge_{\alpha^P}^{\beta^P} \mu_B^P) \leq (\mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_B^P)$ and $(\mu_A^N \vee_{\alpha^N}^{\beta^N} \mu_B^N) \geq (\mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_B^N)$.

Proof. Let a bipolar fuzzy bi-ideal be A=(μ_A^P,μ_A^N) and a bipolar fuzzy left ideal be B=(μ_B^P,μ_B^N) with thresholds[(α^P,α^N),(β^P,β^N)] of R. Since R is regular, so for any a ∈ R there exists x ∈ R such that a = axa. Now,

$(\mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_B^P)(\mathrm{a})$

$=(\sup_{x=\sum_{i=1}^{n} a_i b_i} \bigwedge_{i=1}^{n} \mu_A^P(a_i) \wedge \mu_B^P(b_i)) \wedge \beta^P \vee \alpha^P$

$\geq \mu_A^P(\mathrm{a}) \wedge \mu_B^P(\mathrm{xa}) \wedge \beta^P \vee \alpha^P$

$= \mu_A^P(\mathrm{a}) \wedge (\mu_B^P(\mathrm{xa}) \vee \alpha^P) \wedge \beta^P \vee \alpha^P$

$\geq \mu_A^P(\mathrm{a}) \wedge (\mu_B^P(\mathrm{a}) \wedge \beta^P) \wedge \beta^P \vee \alpha^P$

$=\mu_A^P$(a)$\wedge$(μ_B^P(a)$\wedge\beta^P\vee\alpha^P$=($\mu_A^P \wedge_{\alpha^P}^{\beta^P} \mu_B^P$)(a)

and ($\mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_B^N$)(a)

$=(\inf_{x=\sum_{i=1}^n a_i b_i} \bigvee_{i=1}^n \mu_A^N(a_i) \vee \mu_B^N(b_i)) \vee \beta^N \wedge \alpha^N$

$\leq \mu_A^N$(a)$\vee \mu_B^N$(xa)$\vee \beta^N \wedge \alpha^N$

$=\mu_A^N$(a)$\vee$(μ_B^N(xa)$\wedge \alpha^N$)$\vee \beta^N \wedge \alpha^N$

$\leq \mu_A^N$(a)$\vee$(μ_B^N(a)$\vee \beta^N$)$\vee \beta^N \wedge \alpha^N$

$=\mu_A^N$(a)$\vee$(μ_B^N(a)$\vee \beta^N \wedge \alpha^N$=($\mu_A^N \vee_{\alpha^N}^{\beta^N} \mu_B^N$)(a).

Hence, ($\mu_A^P \wedge_{\alpha^P}^{\beta^P} \mu_B^P$) $\leq$ ($\mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_B^P$) and ($\mu_A^N \vee_{\alpha^N}^{\beta^N} \mu_B^N$) $\geq$ ($\mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_B^N$). □

Theorem 8. For both regular and intra-regular semiring R,$(\mu_A^P \wedge \beta^P) \vee \alpha^P = \mu_A^P \circ_{\alpha^P}^{\beta} \mu_A^P$ and $(\mu_A^N \vee \beta^N) \wedge \alpha^N = \mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_A^N$ for every bipolar fuzzy bi-ideal A=(μ_A^P, μ_A^N) with thresholds $[(\alpha^P, \alpha^N), (\beta^P, \beta^N)]$ of R.

Proof. Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy bi-ideal with thresholds $[(\alpha^P, \alpha^N), (\beta^P, \beta^N)]$ of R and x ∈ R. Since R is both regular and intra-regular, there exists x, p_i, p'_i ∈ R such that x = xax and x = $\sum_{i=1}^n p_i\, xxp'_i$. Thus, x = xax = xaxax = xa($\sum_{i=1}^n p_i\, xxp'_i$)ax = $\sum_{i=1}^n (xap_i x)\,(xp'_i ax)$. Now,

$(\mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_A^P)(x) = (\sup_{x=\sum_{i=1}^n a_i b_i} \left\{\bigwedge_{i=1}^n \{\mu_A^P(a_i) \wedge \mu_A^P(b_i)\}\right\}) \wedge \beta^P\} \vee \alpha^P$

$$\geq \left\{\left\{\bigwedge_{i=1}^n \{\mu_A^P(xap_i x) \wedge \mu_A^P(xp'_i ax)\}\right\} \wedge \beta^P\right\} \vee \alpha^P$$

$$= \left\{\left\{\bigwedge_{i=1}^n \{(\mu_A^P(xap_i x) \vee \alpha^P) \wedge (\mu_A^P(xp'_i ax) \vee \alpha^P)\}\right\} \wedge \beta^P\right\} \vee \alpha^P$$

$$\geq \left\{\left\{\bigwedge_{i=1}^n \{(\mu_A^P(x) \wedge \mu_A^P(x) \wedge \beta^P) \wedge (\mu_A^P(x) \wedge \mu_A^P(x) \wedge \beta^P)\}\right\} \wedge \beta^P\right\} \vee \alpha^P = \{\mu_A^P(x) \wedge \beta^P\} \vee \alpha^P$$

and

$(\mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_A^N)(x) = (\inf_{x=\sum_{i=1}^n a_i b_i} \left\{\bigvee_{i=1}^n \{\mu_A^N(a_i) \vee \mu_A^N(b_i)\}\right\}) \vee \beta^N\} \wedge \alpha^N \leq \left\{\left\{\bigvee_{i=1}^n \{\mu_A^N(xap_i x) \vee \mu_A^N(xp'_i ax)\}\right\} \vee \beta^N\right\} \wedge \alpha^N = \left\{\left\{\bigvee_{i=1}^n \{(\mu_A^N(xap_i x) \wedge \alpha^N) \vee (\mu_A^N(xp'_i ax) \wedge \alpha^N)\}\right\} \vee \beta^N\right\} \wedge \alpha^N$

$$\leq \left\{\left\{\bigvee_{i=1}^n \{(\mu_A^N(x) \vee \mu_A^N(x) \vee \beta^N) \vee (\mu_A^N(x) \vee \mu_A^N(x) \vee \beta^N)\}\right\} \vee \beta^N\right\} \wedge \alpha^N = \{\mu_A^N(x) \vee \beta^N\} \wedge \alpha^N.$$

Hence, $(\mu_A^P \wedge \beta^P) \vee \alpha^P = \mu_A^P \circ_{\alpha^P}^{\beta} \mu_A^P$ and $(\mu_A^N \vee \beta^N) \wedge \alpha^N = \mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_A^N$. □

Theorem 9. Let A=(μ_A^P,μ_A^N) and B=(μ_B^P,μ_B^N) be bipolar fuzzy bi-ideals with thresholds[(α^P,α^N),(β^P,β^N)] of R, then for both regular and intra-regular semiring it satisfies $\mu_A^P \wedge_{\alpha^P}^{\beta^P} \mu_B^P \leq \mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_B^P$ and $\mu_A^N \vee_{\alpha^N}^{\beta^N} \mu_B^N \geq \mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_B^N$.

Proof. Let A=(μ_A^P, μ_A^N) and B=(μ_B^P, μ_B^N) are bipolar fuzzy bi-ideals with thresholds[(α^P,α^N),(β^P,β^N)] of R and let x ∈ R. Since R is both regular and intra-regular, there exists elements a, p_i, p'_i ∈ R such that x=xax and x=$\sum_{i=1}^{n} p_i\, xxp'_i$.
Thus, x = xax = xaxax = xa($\sum_{i=1}^{n} p_i\, xxp'_i$)ax=$\sum_{i=1}^{n}(xap_ix)\,(xp'_iax)$.
Now

$$(\mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_B^P) = (\sup_{x=\sum_{i=1}^{n} a_ib_i} \{\bigwedge_{i=1}^{n}\{\mu_A^P(a_i) \wedge \mu_B^P(b_i)\}\}) \wedge \beta^P\} \vee \alpha^P$$

$$\geq \left\{\left\{\bigwedge_{i=1}^{n}\{\mu_A^P(xap_ix) \wedge \mu_B^P(xp'_iax)\}\right\} \wedge \beta^P\right\} \vee \alpha^P$$

$$= \left\{\left\{\bigwedge_{i=1}^{n}\{(\mu_A^P(xap_ix) \vee \alpha^P) \wedge (\mu_B^P(xp'_iax) \vee \alpha^P)\}\right\} \wedge \beta^P\right\} \vee \alpha^P$$

$$\geq \left\{\left\{\bigwedge_{i=1}^{n}\{(\mu_A^P(x) \wedge \mu_A^P(x) \wedge \beta^P) \wedge (\mu_B^P(x) \wedge \mu_B^P(x) \wedge \beta^P)\}\right\} \wedge \beta^P\right\} \vee \alpha^P$$

$$= \mu_A^P(x) \wedge \mu_B^P(x) \wedge \beta^P\} \vee \alpha^P = (\mu_A^P \wedge_{\alpha^P}^{\beta^P} \mu_B^P)(x)$$

and $(\mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_B^N)$

$$= (\inf_{x=\sum_{i=1}^{n} a_ib_i} \{\bigvee_{i=1}^{n}\{\mu_A^N(a_i) \vee \mu_B^N(b_i)\}\}) \vee \beta^N\} \wedge \alpha^N$$

$$\leq \left\{\left\{\bigvee_{i=1}^{n}\{\mu_A^N(xap_ix) \vee \mu_B^N(xp'_iax)\}\right\} \vee \beta^N\right\} \wedge \alpha^N$$

$$= \left\{\left\{\bigvee_{i=1}^{n}\{(\mu_A^N(xap_ix) \wedge \alpha^N) \vee (\mu_B^N(xp'_iax) \wedge \alpha^N)\}\right\} \vee \beta^N\right\} \wedge \alpha^N$$

$$\leq \left\{\left\{\bigwedge_{i=1}^{n}\{(\mu_A^N(x) \vee \mu_A^N(x) \vee \beta^N) \vee (\mu_B^N(x) \vee \mu_B^N(x) \vee \beta^N)\}\right\} \vee \beta^N\right\} \wedge \alpha^N$$

$$= \mu_A^N(x) \vee \mu_B^N(x) \vee \beta^N\} \wedge \alpha^N = (\mu_A^N \vee_{\alpha^N}^{\beta^N} \mu_B^N)(x).$$

Hence, $\mu_A^P \wedge_{\alpha^P}^{\beta^P} \mu_B^P \leq \mu_A^P \circ_{\alpha^P}^{\beta^P} \mu_B^P$ and $\mu_A^N \vee_{\alpha^N}^{\beta^N} \mu_B^N \geq \mu_A^N \circ_{\alpha^N}^{\beta^N} \mu_B^N$.

4. Conclusion:

In this article, we investigate various results of bipolar fuzzy subsemirings and bipolar fuzzy ideals of a semiring, and also bipolar fuzzy subsemirings and ideals with thresholds are defined, and various results are investigated utilizing the properties of fuzzy ideals with thresholds.

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