IDENTIFYING REPEATING PATHS WITH CRITICAL VALUES, ANALYZING TIME SERIES DATA, CALCULATING THE LYAPUNOV EXPONENT AND MANAGING CHAOS IN CHAOTIC SYSTEMS

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Abstract: Period doubling bifurcations have been observed in many different dynamical systems, both dissipative and conservative, since Feigenbaum's first discovery. The theory behind this phenomenon has been widely studied. In this paper, we investigate three characteristics of a nonlinear chaotic system described by the equation $\emptyset\,(u) = (p-1)u - 2u^3$, where p represents the control parameter as given below:
1) We first analyze the range of the function and establish a general pathway from the stable system to the chaotic region by applying Feigenbaum's theory of period doubling bifurcations.
2) We determine the accumulation point at p = 3.2146537697423 and the Feigenbaum constant δ = 4.669138913547. Additionally, we use time series analysis and the Lyapunov exponent to validate our results regarding the bifurcation points and chaotic behavior.
3) Various scientific methods are used to demonstrate how chaos can be controlled.
4) Finally, we pose several interesting open questions for future research.

Keywords: Feigenbaum scaling: Period-doubling transition; Lyapunov characteristic exponent; Chaos control.

1. Introduction:

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Here we first briefly explain the basic concept of Chaos. Chaos actually represents a natural phenomenon that disrupts the stability of systems. It has intrigued both mathematicians and physicists since the late 19th century. Henri Poincare, a French mathematician, was a pioneer in this field, introducing a novel analytical approach known as the geometry of dynamics. In certain nonlinear discrete models, period doubling bifurcation is observed as a pathway to chaotic behavior. The logistic map, a simple quadratic model, exhibits complex dynamics through this bifurcation process. In 1964, Lorenz demonstrated in his paper that the logistic map could also exhibit non-periodic behavior. In the early 1970s, R.M May investigated the period-doubling behavior of the logistic map but did not examine the dynamics beyond the accumulation point. Our study is primarily based on the theory developed by Feigenbaum. A few key concepts are highlighted in the following system, [1,2,5,9,21]

1.1 Orbit: Given a function ∅ :U→U and an element u ∈ U the orbit of u under ∅ is the sequence {u, ∅ (u), $\emptyset^{2}$(u),…… $\emptyset^{n}$(u)} which is denoted as O(u)

1.2 Fixed Points: A fixed point of a function ∅ is an element u∗ that satisfies ∅(u∗) = u∗. This implies that $\emptyset^{n}$(u∗) = u∗ for any positive integer n, meaning the orbit of u∗ is consistently {u∗, u∗,,……,u∗}.

1.3 Periodic Points and Periodic Orbits: An orbit is termed periodic if it eventually returns to its starting point. Specifically, u∗ is a periodic point with period n if $\emptyset^{n}$(u∗)=u∗ for the smallest positive integer n. The corresponding orbit is called its periodic orbit.

1.4 Stability of a Fixed Point: Consider a fixed-point u such that $\emptyset$ (u)=u. If we slightly perturb the orbit by a small amount, ϵ_n such that u_n= u+ϵ_n and expand this perturbation to the first order, we get: u+ϵ_{n+1}=$\emptyset$(u+ϵ_n)

$$\approx \emptyset(u)+\acute{\emptyset}(u)\epsilon_n$$
$$\epsilon_{n+1} \approx \acute{\emptyset}(u)\epsilon_n$$

From the given the relation , we can determine the stability of a fixed point as follows:
A fixed point is considered stable if the derivative $\acute{\emptyset}(u)$ is less than 1, i,e if $\acute{\emptyset}(u) < 1$
and unstable if the derivative $\acute{\emptyset}(u)$ is greater than 1, i,e if $\acute{\emptyset}(u) > 1$.

1.5 Bifurcation:
A bifurcation happens when a system splits into two different paths or outcomes. In dynamical systems, it often refers to a process where the system starts showing more complex behavior as you change a certain parameter. For example, if a system initially repeats every step (period 1), it might start repeating every two steps (period 2), then every four steps (period 4), and so on, eventually leading to chaotic behavior. This process of doubling the repeating cycle is known as period doubling and is a common way for systems to become chaotic, [3,4,11,12, 17]

1.6 Feigenbaum Delta:
Feigenbaum delta is a special number that helps predict how a system changes as it goes from a simple repeating cycle to a more complex one. This number, about 4.669, was discovered by Mitchell Feigenbaum in the 1970s. It measures the ratio between the distances of certain points where the system's behavior changes, helping us understand how a system transitions from one type of repeating pattern to another. So the value of δ can be determined as,
$\delta = \frac{p_n - p_{n-1}}{p_{n+1} - p_n}$= 4.669…… where p_n is the value of bifurcation for each n.

1.7 Accumulation Point:
An accumulation point is a specific value of a system's parameter beyond which the system starts to behave chaotically. It marks the point where the system transitions from having predictable behavior to unpredictable and complex behavior.

2. Main Results: [4, 8, 18, 19, 20]
2.1 Analysis of stability using linear method:

Let us consider the system $\emptyset(u) = (p-1)u - 2u^3$ as our model for studying various dynamical properties. Here p is the controlling parameter.

$$\text{Now} \quad \phi'(u) = 0$$
$$\Rightarrow (p-1) - 6u^2 = 0$$
$$\Rightarrow u^2 = \frac{p-1}{6}$$
$$\Rightarrow u = \left(\frac{p-1}{6}\right)^{\frac{1}{2}}$$
$$\text{Again} \quad \phi''(u) = -12u$$

So at $u = \left(\frac{p-1}{6}\right)^{\frac{1}{2}}$, $\phi''(u) = -12\left(\frac{p-1}{6}\right)^{\frac{1}{2}}$. This value is negative for which the value $4\left(\frac{p-1}{6}\right)^{\frac{3}{2}}$ as the maximum value of the function $\phi(u)$. We can take the range of $\phi(u)$ as $\left]-\infty, 4\left(\frac{p-1}{6}\right)^{\frac{3}{2}}\right]$.When $\phi(u) = u$, we obtain $u_1 = 0$, $u_2 = -\left(\frac{p-2}{2}\right)^{\frac{1}{2}}$, $u_3 = \left(\frac{p-2}{2}\right)^{1/2}$ as the fixed point of the given system. Now applying the stable fixed-point definition, we obtain

(i) At $u_1 = 0$, we get $\left|\frac{d\phi}{du}\right| = |p-1|$.
So u_1 is stable if $|p-1| < 1 \Rightarrow -1 < p-1 < 1 \Rightarrow 0 < p < 2$

(ii) At $u_2 = -\left(\frac{p-2}{2}\right)^{\frac{1}{2}}$, we get $\left|\frac{d\phi}{du}\right| = |2p-5|$.
So u_2 is stable if $|2p-5| < 1 \Rightarrow -1 < 2p-5 < 1 \Rightarrow 4 < 2p < 6 \Rightarrow 2 < p < 3$

(ii) At $u_3 = \left(\frac{p-2}{2}\right)^{\frac{1}{2}}$, we get the same value of $\left|\frac{d\phi}{du}\right|$ as u_2.
Now we have noticed that the fixed point $u_1 = 0$ is stable fixed point for $0 < p < 2$ and rest of the others are unstable for the same definite region. Hence for the definite region we can find the trajectories with the interval $0 < u < 1$ approaching the fixed point at $u_1 = 0$.
Within the range $2 < p < 3$, the picture has changed to different end. Within the range, the fixed point $u_1 = 0$ becomes unstable fixed point with the trajectories moved to the other fixed points.

In this way when p crosses the value 3 we obtain $\left|\frac{d\varphi}{du}\right| > 1$ for each fixed points and thus the equilibrium points of the model are not stable. So the first bifurcation value of the model is 3.

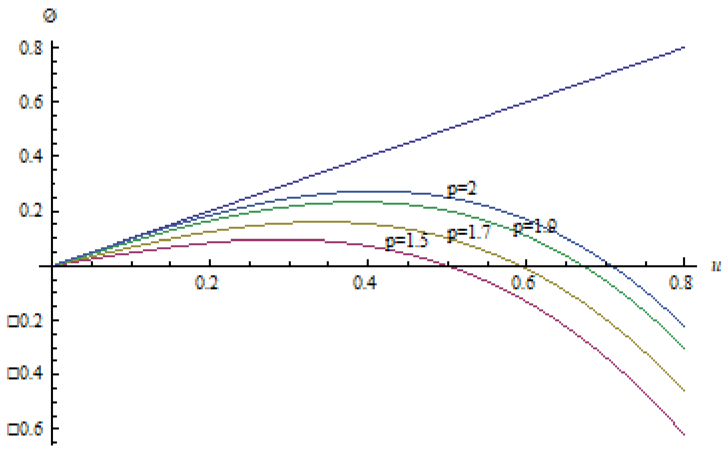

Fig 1.1: Trajectories for 0 < p < 2 (intersection of the model and $\emptyset(u) = u$)

In the range where 2< p < 3, the situation has changed. Here, the point $u_1 = 0$ becomes an unstable fixed point, meaning that trajectories are now drawn toward the other fixed points. In other words, as we adjust the control parameter, the stability of the fixed points switches.

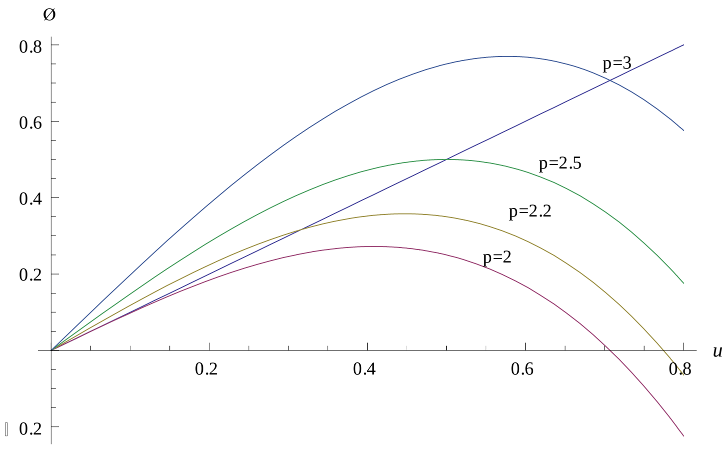

Fig 1.2: Trajectories for 2< p < 3 (intersection of the model and $\emptyset(u) = u$)

When the value of p surpasses 3, the absolute value of the derivative $\left|\frac{d\phi}{du}\right|$ becomes greater than 1 at each fixed point. This indicates that the stability of the fixed points in the model is disrupted. Thus, the first bifurcation point of the model is approximately 3

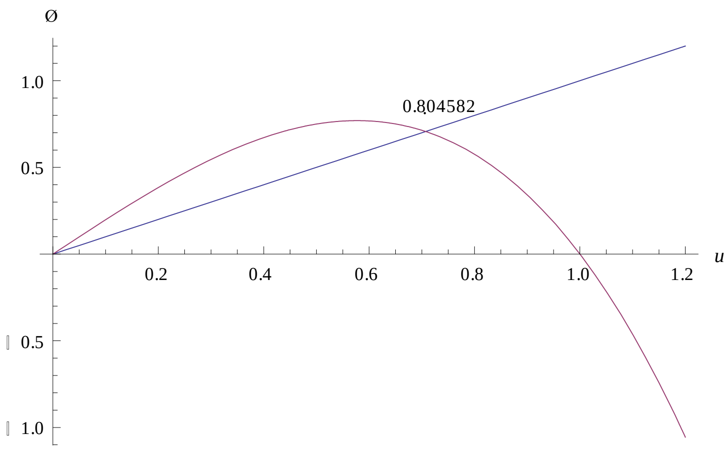

Fig 1.3: Intersection of the model and $\emptyset(u) = u$ at p=3.

Next, we look at periodic points with a period of two or more. We do this by solving the equation $\emptyset^2(\mathrm{u}) = \mathrm{u}$

$$\Rightarrow \emptyset(((\mathrm{p}-1)\mathrm{u} - 2\mathrm{u}^3)) = \mathrm{u}$$

$$\Rightarrow (\mathrm{p}-1)\,\emptyset(\mathrm{u}) - 2\emptyset(\mathrm{u})^3 = \mathrm{u}$$

$$\Rightarrow (\mathrm{p}-1)\{(\mathrm{p}-1) - 2\mathrm{u}^3\} - 2\{(\mathrm{p}-1) - 2\mathrm{u}^3\}^3 = \mathrm{u}$$

$$\Rightarrow \mathrm{u} - 2\mathrm{pu} + \mathrm{p}^2\mathrm{u} + 4\mathrm{u}^3 - 8\mathrm{pu}^3 + 6\mathrm{p}^2\mathrm{u}^3 - 2\mathrm{p}^3\mathrm{u}^3 + 12\mathrm{u}^5 - 24\mathrm{pu}^5 + 12\mathrm{p}^2\mathrm{u}^5 + 24\mathrm{u}^7 - 24\mathrm{pu}^7 + 16\mathrm{u}^9 == \mathrm{u}$$

Now using “Mathematica”, we solve the equation and get the solution as follows:

$$\{u_1 \to 0\},$$

$$\left\{u_2 \to -\frac{\sqrt{-2+p}}{\sqrt{2}}\right\},$$

$$\left\{u_3 \to \frac{\sqrt{-2+p}}{\sqrt{2}}\right\},$$

$$\left\{u_4 \to -\frac{\sqrt{p}}{\sqrt{2}}\right\},$$

$$\left\{u_5 \to \frac{\sqrt{p}}{\sqrt{2}}\right\},$$

$$\left\{u_6 \to -\frac{1}{2}\sqrt{-1+p-\sqrt{-3-2p+p^2}}\right\},$$

$$\left\{u_7 \to \frac{1}{2}\sqrt{-1+p-\sqrt{-3-2p+p^2}}\right\},$$

$$\left\{u_8 \to -\sqrt{-\frac{1}{4}+\frac{p}{4}+\frac{1}{4}\sqrt{-3-2p+p^2}}\right\},$$

$$\{u_9 \to \sqrt{-\frac{1}{4}+\frac{p}{4}+\frac{1}{4}\sqrt{-3-2p+p^2}}\}\}$$

We shall now determine the derivative of $\emptyset^2(u)$ as follows:

$$\frac{\mathbf{d^2 \emptyset^2(u)}}{\mathbf{du^2}} = \mathbf{(-1+p)^2 - 6(-1+p)u^2 + 6(1-p+6u^2)(u-pu+2u^3)^{\ 2}}$$

Here, the variations in the derivative values of the original map and its second iteration have been noted. When the value of p increases through 3, the value $\frac{d\phi}{du}$ passes through -1. Now we have the identical derivative of $\emptyset^2(u)$ for the fixed points u_9 , u_8, u_7, u_6 which is $7+4p-2p^2$. Again, we have the identical derivative of $\emptyset^2(u)$ for the fixed points u_5 and u_4 which is $1+1+4p+4p^2$ and for the fixed points u_3, u_2 its value is $25-20p+4p^2$. Lastly the derivative of $\emptyset^2(u)$ is found to be 1-2p+p^2 i, e $(-1+p)^2$. As the value of p goes above 3, the behavior of the system changes, causing the values to become unstable. Below 3, the function settles on a single value of u while just above this threshold, the system starts to oscillate between two values, u_8 and u_6.

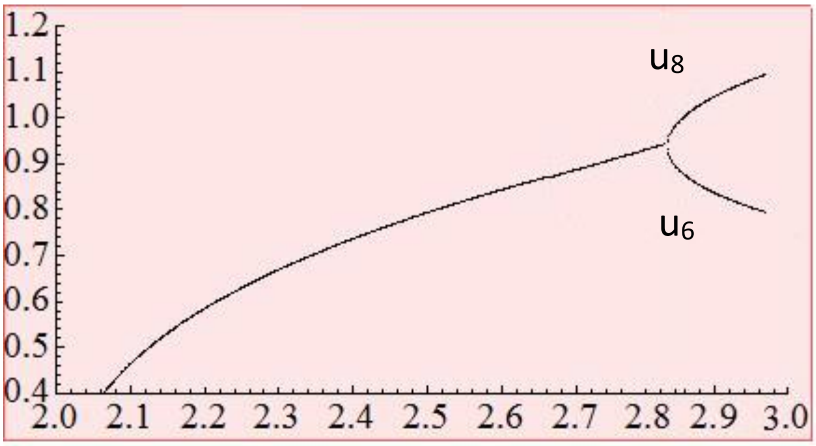

Fig 1.4: Exchange of the map between two fixed points

When p increases, the derivative $\emptyset^2(u)$ decreases, which stabilizes the fixed points. This stability lasts until p reaches approximately 3.1614. At this point, the derivative of $\emptyset^2(u)$ calculated at the two-cycle fixed points becomes -1. As p continues to rise, this derivative becomes even more negative, which makes the fixed points unstable. Thus, the second bifurcation point occurs at $p = p_2 \approx 3.1614$. After this, the system starts to alternate between four fixed points.

To find the bifurcation point for higher periods, we use the function $I(p_n) = \frac{d^n u}{dx^n}$. By calculating the bifurcation points for period one p_1 and period two p_2 manually, we can find the bifurcation point for period three p_3 using the bisection method, and so on. We utilize both the Newton-Raphson method and the bisection method to identify the bifurcation points for period four and higher. We also create a C program to determine these bifurcation points and the experimental value of the Feigenbaum delta.

Table 1.1: Calculation of bifurcation points and the Feigenbaum constant

Periods	One of the periodic points	Bifurcation values	Feigenbaum delta $\delta = \frac{p_n - p_{n-1}}{p_{n+1} - pn_n}$
1	0.804582	3	
2	1.025933	3.1614	
4	1.02735400	3.204635086273	4.41856022780
8	1.041645407697	3.212504592287	4.612549128072
16	1.03729335598	3.214193334878	4.638279594249
32	1.041417401572	3.214555151631	4.65439418808
64	1.050166132090	3.214632648454	4.663454304524
128	1.042256081010	3.214649246202	4.666546792005
256	1.04798842756937	3.216528000945	4.662343687382
512	1.045438990531	3.214653562262	4.6635021853407
1024	1.048642532948	3.214653725313	4.66543552829
2048	1.042176548168	3.214653760234	4.669138913547

From the above table we can establish the Feigenbaum delta up to 4.669138913. We shall now draw some graphs which show the periodic points in respective periods.

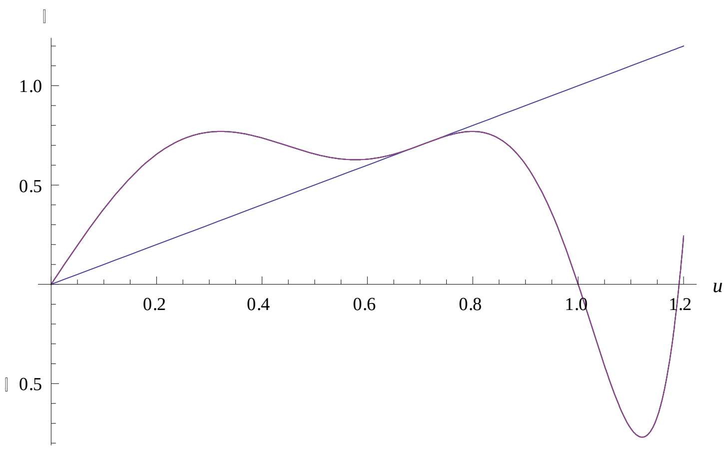

Fig 1.5: Graph of $\emptyset^2$(u) and $\emptyset(u) = u$

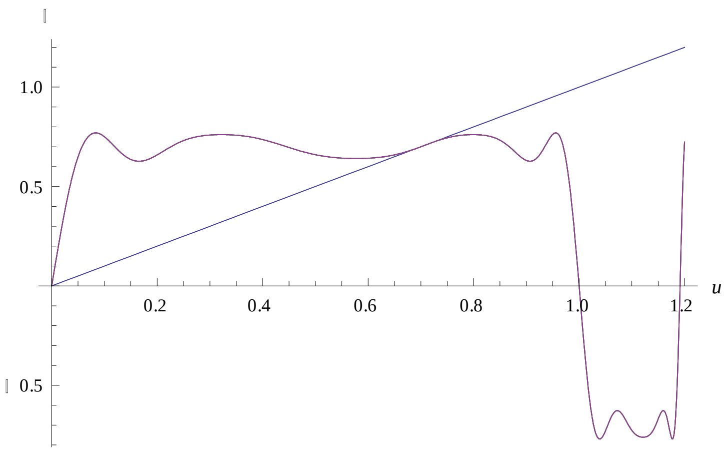

Fig 1.6: Graph of $\emptyset^4$(u) and $\emptyset(u) = u$

The bifurcation diagram of our model, showing the fixed-point positions against the parameter p, demonstrates the common path to chaos in our system.

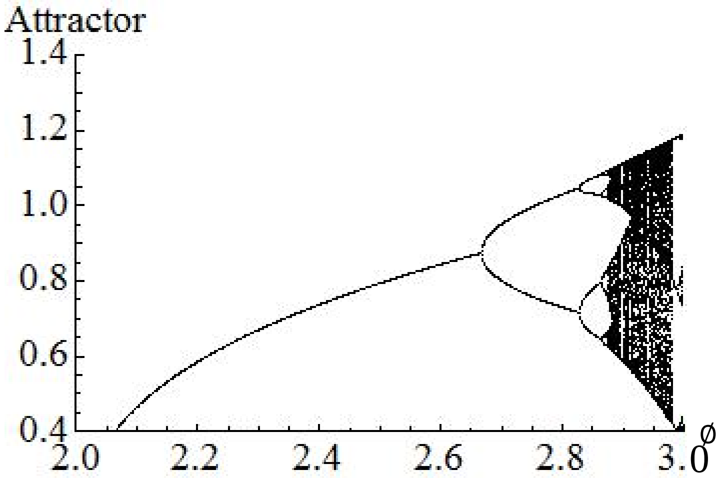

Fig 1.7: bifurcation diagram

2.2: Accumulation Point:

With the help of the experimental bifurcation points, the sequence of accumulation points $\{p_{\infty,n}\}$ is determined using the following formula.

$$p_{\infty,n} = \frac{p_{n+1} - p_n}{\delta - 1} + p_{n+1}$$

The accumulation points for different values of n are,

$$p_{\infty,1} = 3.212504392257$$
$$p_{\infty,2} = 3.214570788834$$
$$p_{\infty,3} = 3.214649397765$$
$$p_{\infty,4} = 3.2146537634546$$
$$p_{\infty,5} = 3.214653763484$$
$$p_{\infty,6} = 3.2146537701761$$
$$p_{\infty,7} = 3.2174653769923$$
$$p_{\infty,8} = 3.2165289782498$$
$$p_{\infty,9} = 3.2146537914572$$
$$p_{\infty,10} = 3.21746537697423$$
$$p_{\infty,11} = 3.2146537697423$$

The above sequence converges to the value 3.2146537697423 which is the required accumulation point beyond which chaos occurs.

2.3: Evaluation of Time Series:[6,7,11,12,15,16]

Here, we aim to quantify chaos through time series analysis. By examining data over time, we can track the changes that have occurred in the past, providing valuable insights into the system's behavior. This method is particularly useful for predicting future dynamics [1, 3, 6, 16].

To demonstrate this, a series of time series experiments are conducted, all starting from the same initial value of 0.8 but varying the parameter values. The time series graphs below display the results for different parameter settings. The horizontal axis represents the number of iterations, while the vertical axis shows the values obtained after each iteration.

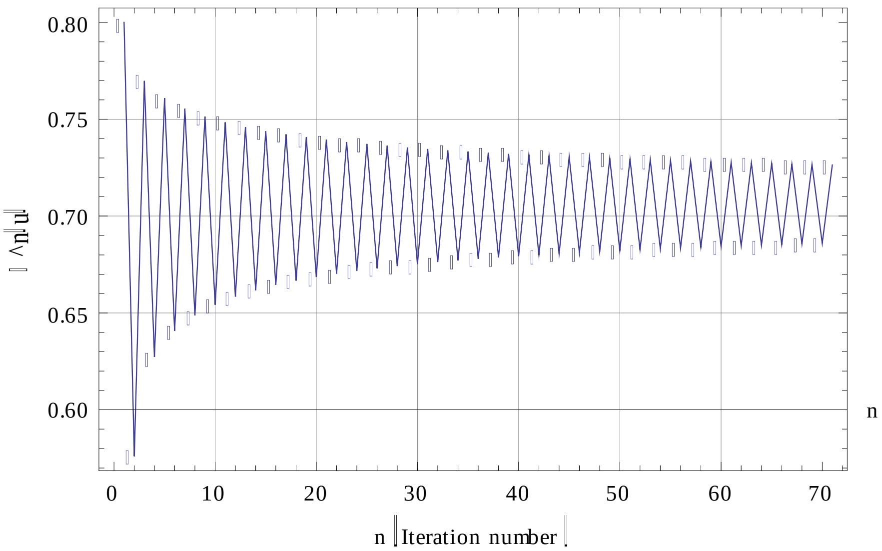

Fig 1.8: Time Series graph of Period 1 with p=3,

After a few iterations here, the points are connected by a line segment. The time series graph shows a stable and non-sensitive behavior, where the system eventually reaches the same final state at a fixed point.

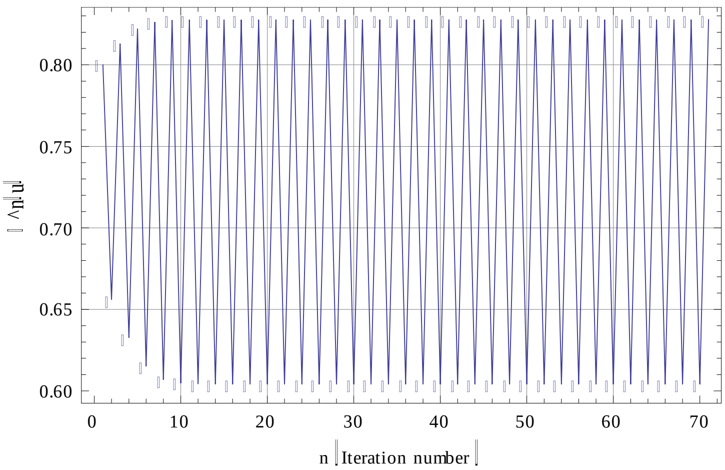

Fig 1.9: Time Series graph of Period 2 with p=3.16

Here also, after several iterations, we observe a periodic pattern, where the system alternates between two fixed points with a constant amplitude, and the cycle repeats. Similarly, oscillations between 8 fixed points, 16 fixed points, 32 fixed points, 64 fixed points, and so on can also be seen.

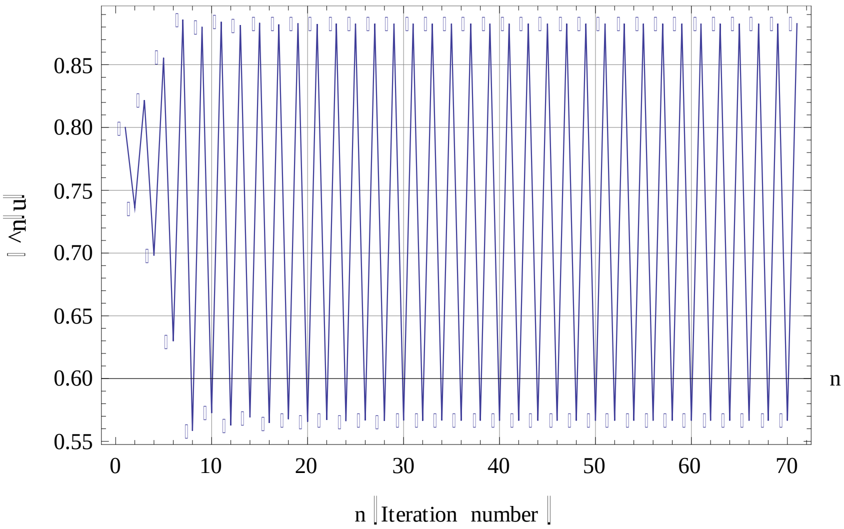

Fig 2.1: Time Series graph of Period 4 with p=3.2

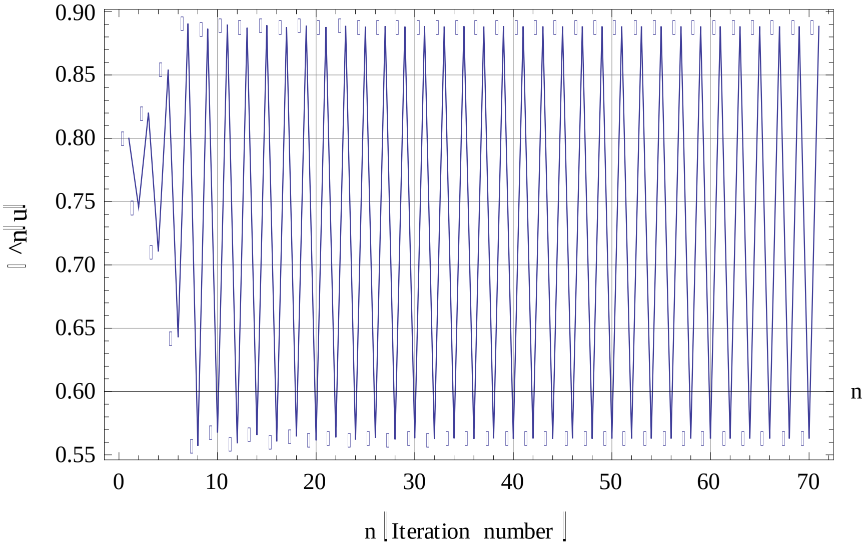

Fig 2.2: Time Series graph of Period 8 with p=3.212

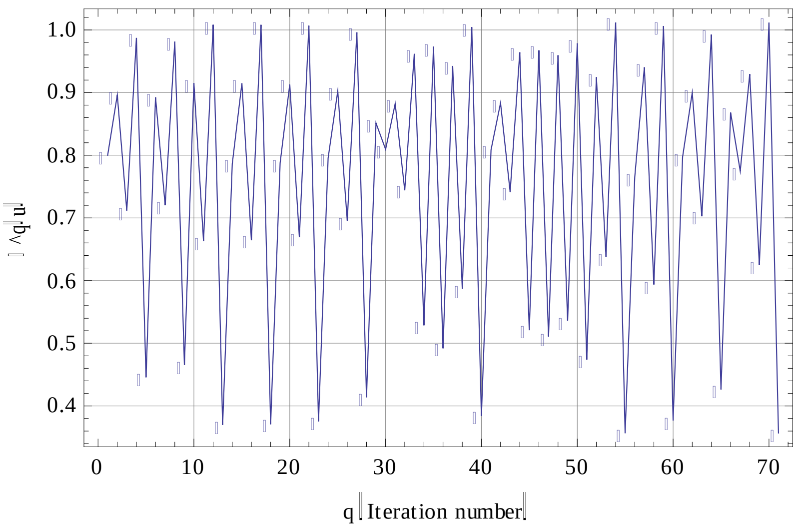

Fig 2.3: Time series graph for the chaotic behavior at p=3.4

Here we have seen irregular behavior at p=3.4. So, we can conclude that at 3.4, our model starts indicating chaotic behavior, a point inside the chaotic region.

2.4 Lyapunov Exponent: [2,5,6,19,21]

The Lyapunov exponent is a way to measure how chaotic a system is. It helps us understand the behavior of dynamic systems, especially in nonlinear systems, where chaos causes randomness and the loss of information, making the system's behavior more complex. The Lyapunov exponent measures how fast the orbits (trajectories) of two nearby points diverge or converge after many iterations. For a system described by the equation $u_{n+1}=\emptyset(u_n)$, we start with two initial points u_0 and $u_0+\varepsilon$ and observe how the distance between them changes after several iterations.

We have now,

$$d_n \equiv \left|\emptyset^{(n)}(u_0+\varepsilon)-\emptyset^{(n)}(u_0)\right|$$

If the behavior is unpredictable, this distance increases rapidly as n increases.

So, we have

$$\frac{d_n}{\varepsilon}=\frac{\left|\emptyset^{(n)}(u_0+\varepsilon)-\emptyset^{(n)}(u_0\right|}{\varepsilon}\equiv e^{n\lambda(u_0)}$$

$i.e.,\ \lambda(u_0)=\frac{1}{n}ln\left|\frac{\emptyset^{(n)}(u_0+\varepsilon)-\emptyset^{(n)}(u_0}{\varepsilon}\right|$, where λ is the associated Lyapunov exponent. Let us take limit $\varepsilon\to 0$, with the application of the chain rule of differentiation we obtain

$$\lambda(u_0)=\frac{1}{n}\ln\left(|\emptyset'(u_0)||\emptyset'(u_1)|\ldots\ldots\ldots|\emptyset'(u_{n-1})|\right)$$

$$=\frac{1}{n}\left(\ln|\emptyset'(u_0)|+ln|\emptyset'(u_1)|+\cdots\ldots\ldots+ln|\emptyset'(u_{n-1})|\right)$$

$$\lambda(u_0)=\frac{1}{n}\sum_{i=0}^{n-1}ln|\emptyset'(u_i)|$$

lastly taking the limit $n\to\infty$, we get the expression for Lyapunov exponent $\lambda(u_0)$ is

$$\lambda(u_0)=\lim_{n\to\infty}\frac{1}{n}\log\left|\frac{d\emptyset^n(u_0)}{du}\right|$$

Now, when $\lambda<0$ we obtain, $\lim_{n\to\infty}\emptyset^{(n)}(u_0)=\lim_{n\to\infty}\emptyset^{(n)}(u_0+\varepsilon)$

This means, the system's behavior is not influenced by the initial value. Again, when $\lambda>0$, we get $\left|\emptyset^{(n)}(u_0+\varepsilon)-\emptyset^{(n)}(u_0)\right|$ and the system shows exponential growth, leading to divergence and indicating chaotic behavior. A positive Lyapunov exponent signifies chaos. In summary:

If $\lambda<0$, the system exhibits a periodic solution. If $\lambda>0$, the system displays chaotic behavior and if $\lambda=0$, it corresponds to a bifurcation point.

The Lyapunov exponent is calculated by averaging the logarithm of the derivative of the function at each iteration point. Mathematically, it is given as

$$\text{Lyapunov exponent}(\mu)=1/n\sum_{i=0}^{n}\log|\,\emptyset(u_i)|.$$

To compute this, we start with the initial value u_0 = 0.8 and a nearby point $u_0+\varepsilon$, using a program in 'Mathematica'. The Lyapunov exponents are then calculated for values of the parameter 2 < p < 4. The table below presents the Lyapunov exponent values near our accumulation point.

Table 1.2: value of Lyapunov exponent near accumulation point

Parameter	Lyapunov exponent
3.212	−0.0831565668185093
3.213	−0.26317485157623466
3.214	−0.0863262897967682
3.215	0.17056495686426428

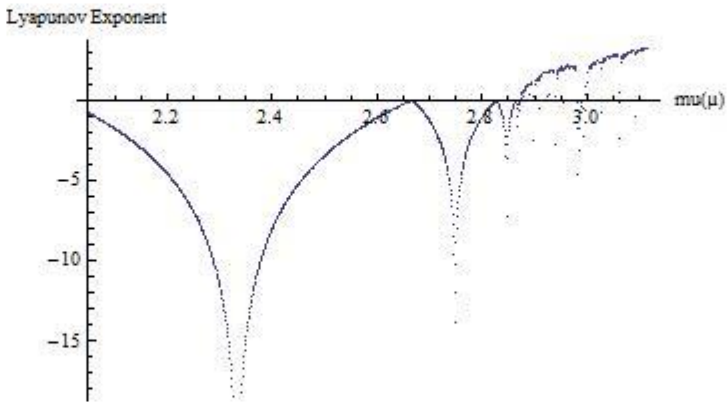

Fig 2.4: The graph representing the Lyapunov exponent of our model

The graph of the Lyapunov exponent confirms the accumulation point of our model. It shows that part of the graph lies on the negative side of the parameter, indicating regular behavior, while the portion on the positive side confirms the presence of chaos in our model.

2.5 Controlling of Chaos in our model and in discrete dynamical Systems: [1,9,13,15,16,17]

The natural world often behaves in a chaotic way, and humans have a strong desire to bring order to it. In our model, we observe a process called period doubling bifurcation, which leads to chaos. As the value of p increases, the system becomes more chaotic, and its long-term behavior becomes unpredictable, especially when considering periodic points. This is why controlling chaos is important.

In a chaotic system, there are many unstable periodic orbits and a strange attractor. To control chaos, we need to stabilize some of these unstable orbits, which can help bring the system to a more regular and predictable state.

The OGY method, proposed by Ott, Grebogi, and Yourke in 1990, was the first technique for controlling chaos. In discrete dynamics, Martiaz and Guemez introduced the instantaneous pulse method to control chaos, and Chua modified it in 1997. In this paper, we use a technique called periodic proportional pulses to control unstable periodic orbits in the strange attractors of our model. Again, in discrete dynamics, Chau discussed the methods of Martiaz and Guemez using a simple one-dimensional chaotic mode

$u_{n+1}=\emptyset(u_n)$, where $\emptyset$ is a function of I on to itself, I is an interval.

A new function $\varphi(u)$ is introduced as:

$$\varphi(u) = \lambda\emptyset^q(u),$$

where λ is a factor that modifies the function every q iteration.

To find the fixed points of $\varphi(u)$, we solve the equation $\lambda\emptyset^q(u)=u$ and the fixed points are stable if the following condition holds: That is if $\left|\frac{d\lambda\emptyset^q(u_i)}{du}\right| < 1$.

It turns out that the stable fixed points of $\varphi(u)$ correspond to stable periodic points of the original function $\emptyset$, with a period of q.

One more function is introduced as follows,

$$K_q(u) = \frac{u}{\emptyset^q(u)}\frac{d\lambda\emptyset^q(u)}{du}$$

From the above equations, $|K_q(u_i)| < 1$, if this condition is satisfied for a fixed point u_i, then with the jolt factor λ, we can stabilize the chaotic periodic orbits that pass through this point. By using these techniques, we aim to control chaos in our model. In our model, we have seen chaotic behavior when it passes through the period doubling cascade.

After the accumulation point 3.2146537697423 of period doubling cascade the chaotic behavior exists. So, picking the parameter value p=3.4 for our model we shall proceed now.

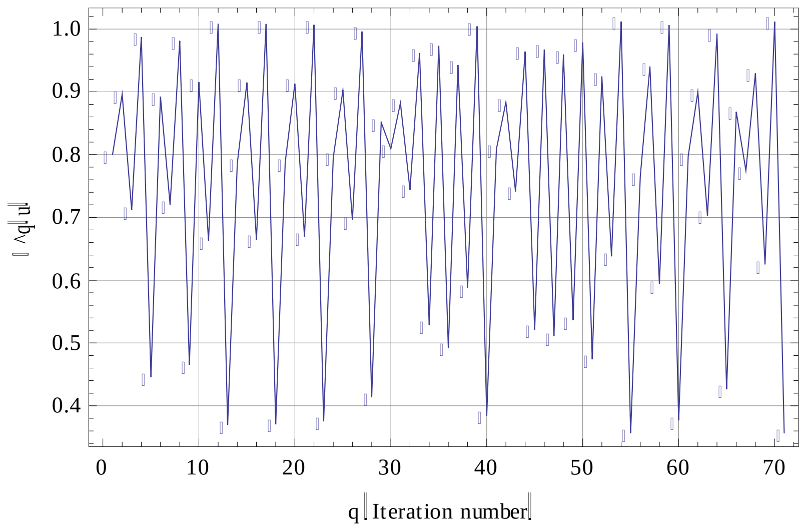

Fig 2.5: Time series graph at p=3.4 q =1 and p= 3.4, the control curve is drawn in the range -1< $K_q(u_i)$ < 1

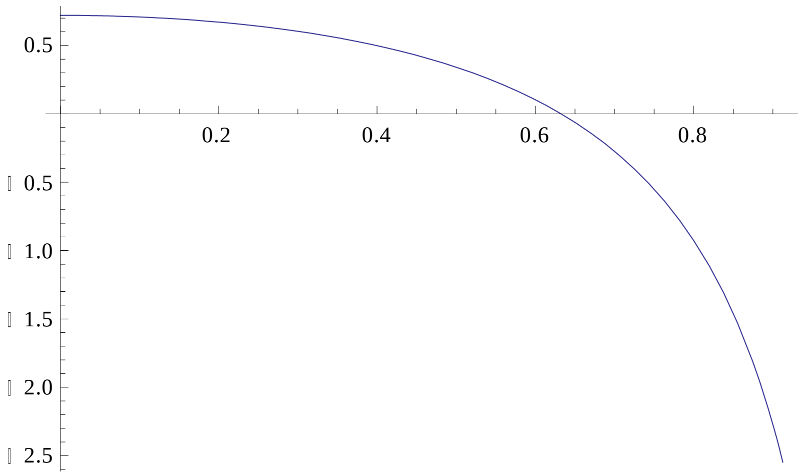

Fig 2.6: Control Curve for the Parameter b=3.4 and q=1

We find that the range for u_i is from 0 to 0.912581, which is determined by solving the equations $K_q(u_i)$ = -1 and $K_q(u_i)$ = 1 in 'Mathematica'. When we set u_i= 0.8, we calculate the jolting factor λ (lambda) to be approximately 0.7204610951008648. The control procedure then stabilizes the chaotic periodic orbit as given below.

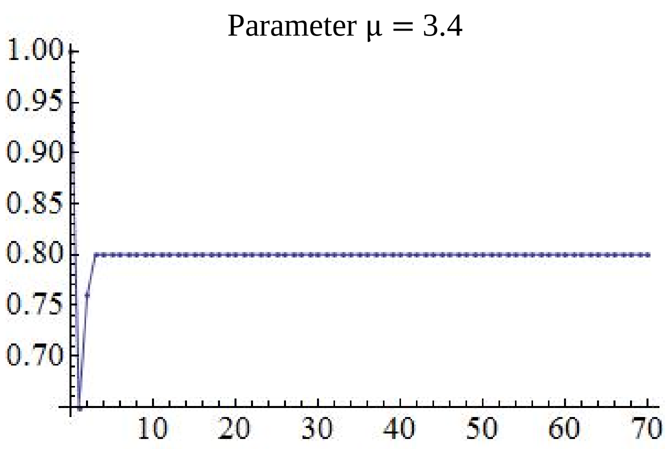

Fig 2.7 : p= 3.4, q=1 and λ = 0.7204610951008648

The control graph in the same range for q=2 and p=3.4 is given below.

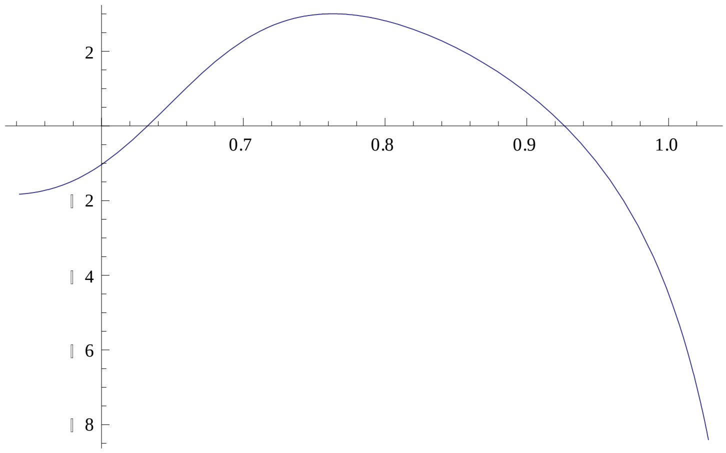

Fig 2.8 : Control graph at q=2, p=3.4

The range of u_i here is between 0.5420923411206577 and 1.028167901028882. If we choose u=1.01, we get λ=0.6115953003708272. With this value of λ , the control method stabilizes the chaotic orbits that pass through these periodic points, bringing them to a stable orbit with a period of one.

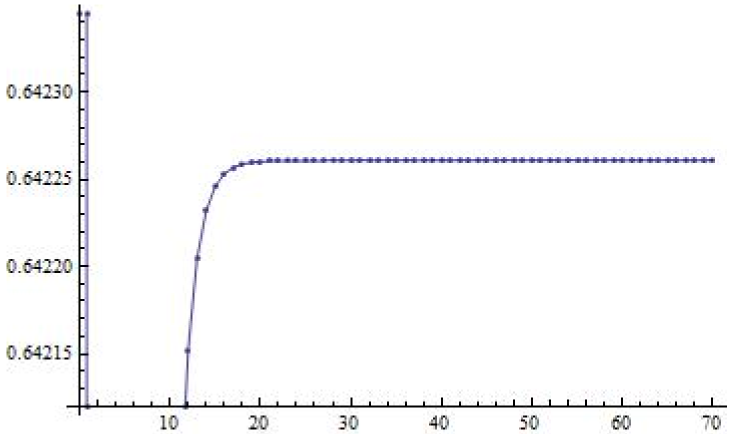

Fig 2.9: p=3.4, q=2, λ =0.6115953003708272

In the same way, when q=3, q=4 at p=3.4 the range of u_i's are (0.29123078320996315, 1.1487906778431043) and (0.153751,1.15225) respectively.

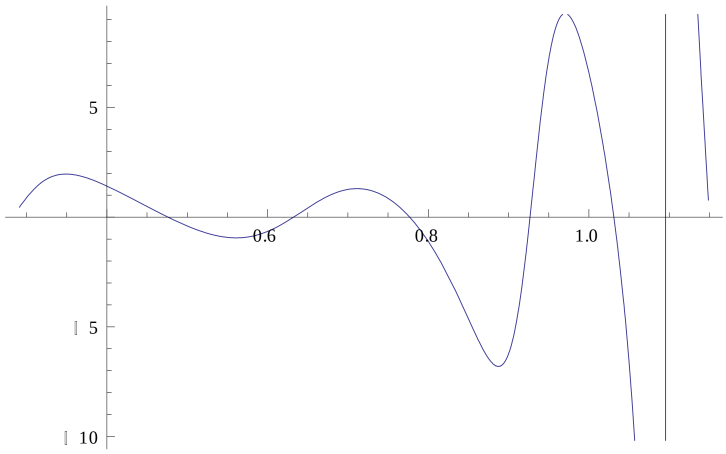

Fig 3.1: Control curve for q=3,p=3.4

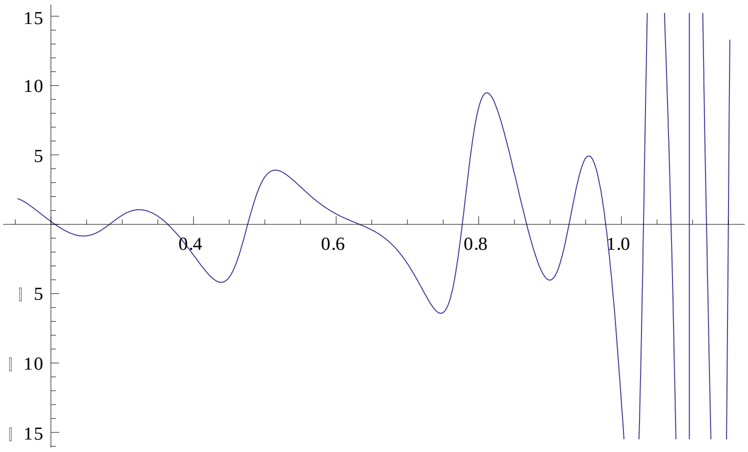

Fig 3. 2: Control curve for q=4, p=3.4

When we take $u_i = 1.1$ for q=3 we get $\lambda = 0.5301838077079213$,
Considering$x_i = 1.14$ for q=4, we get $\lambda = 0.5301838077079213$

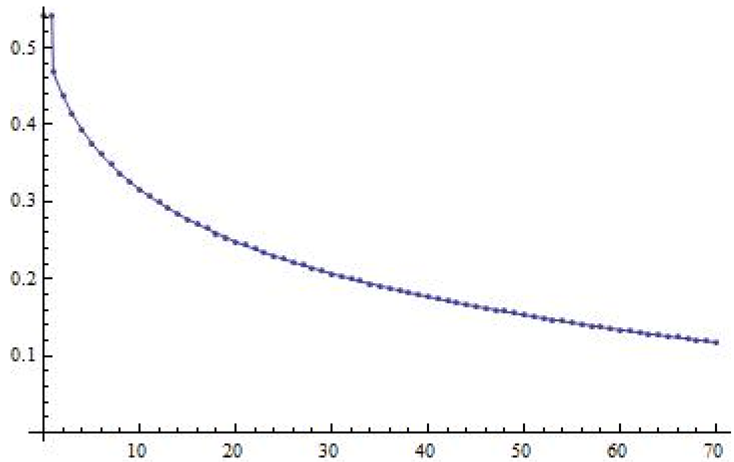

Fig 3.3: p=3.4, q=3 and λ=0.5301838077079213

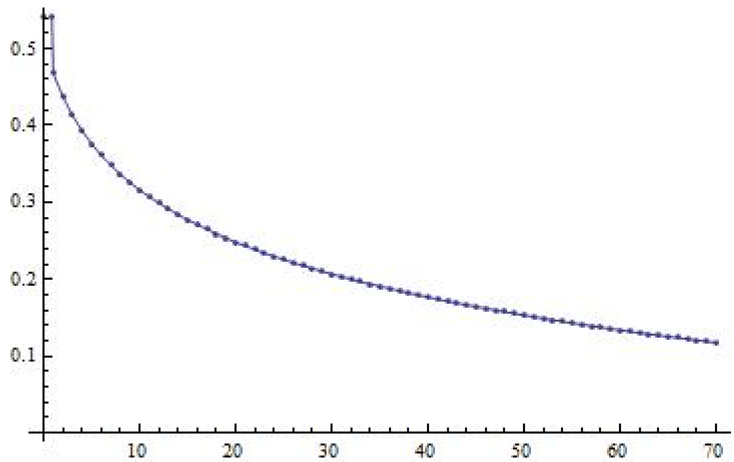

Fig 3.4: p=3.4, q=4 and λ=0.5301838077079213

At the end we see that the chaotic behavior of this model can be controlled for lower periodic points. However, it becomes challenging to control the chaos when the value of q is higher.

3. Conclusion:
It can be concluded that a cubic map of the form ϕ (u) = (p-1) u – $2u^3$ shows period doubling bifurcations that lead to chaos. Bifurcation values are calculated for periods 2, 4, 16, 256, and so on, along with the Feigenbaum constant delta. An accumulation point is found beyond which chaotic behavior is observed in the time series analysis. Chaos is confirmed by calculating Lyapunov exponents. Lastly, successful methods for controlling the chaos are explored.
Open Problems: The following issues can be explored:
Consider a polynomial function $\emptyset(u) = s_0 + s_1u + s_2u^2 + \cdots + s_nu^n$
What are the connections between the parameters that lead to chaos?
What are the conditions under which chaos does not occur?
What are the parameter ranges that allow the evaluation of different fractal dimensions?

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