Γ −DERIVATIONS AND THEIR NORM PROPERTIES IN THE PROJECTIVE TENSOR PRODUCT OF Γ −BANACH ALGEBRAS

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Abstract: The abstract should appear like this. The abstract should appear like this. The abstract should appear like this. The abstract should appear like this. The abstract should appear like this.
This paper deals with many illuminating results on different Γ −derivations in the Projective Tensor product of Γ − Banach algebras. The following results are established here:
Let (B, Γ) be the projective tensor product of two Γ − Banach algebras (B_1, Γ_1) and (B_2, Γ_2). If φ_1 and φ_2 be generalized derivations / generalized Jordan derivations / generalized inner derivations on (B_1, Γ_1) and (B_2, Γ_2) respectively, then there exists a derivation φ on the projective tensor product (B, Γ) having the same properties. Moreover, $\|\varphi\| \leq \|\varphi_1\| + \|\varphi_2\| \leq 2\|\varphi\|$ is a crucial result in this field.

Keywords: Projective tensor product of Γ- Banach algebras; Derivation; Jordan derivation; Inner Derivations

1. Introduction:

In general, there is no natural way of introducing a binary algebraic multiplication in many interesting sets of illuminating objects, viz. the set of all linear transformation $\mathcal{L}(X, Y)$ from a vector space X into another vector space Y, the set of all rectangular matrices of order $m \times n,\ m \neq n$, the set Ci of all purely imaginary complex numbers of the form xi, where, x is real and i = $\sqrt{-1}$, etc. so that they become a ring. To offset this difficulty, various authors like Nobusawa [1964], Barnes [1966], Coppage [1971], Booth [1984] and Bhattacharjya [1989] etc. have developed a ternary system which has ultimately led mathematicians to introduce the notions of a $\Gamma -$ ring and a Γ −Babach algebra. When a good number of prominent mathematicians have been able to extend many deep and profound results from general rings and algebras to Γ − rings, Γ −algebras, Γ − modules, Γ − normed algebras, Γ −Babach algebra, Γ −radicals, Γ −tensor products etc. and more interestingly, when their results have appeared to be the smoothest and most satisfactory theory including bulk of applications in different branches of Mathematics, an innovating and challenging outlook has been evolved and then this field suggests a very wide scope of doing research, [3,4,5,7,10,12]

2. Some Basic Concepts: [1,3,6,8,14,16]

Definition 2.1: Let us consider two additive groups, B = { u, v, w, ……} and
Γ = { $a, b, c, \ldots\ldots$} . Define two mappings:
$\theta : B \times \Gamma \times B \rightarrow B$ and $\varphi : \Gamma \times B \times \Gamma \rightarrow \Gamma$ such that the following three conditions are satisfied:

(i) $\theta(u_1 + u_2, a, v) = \theta(u_1, a, v) + \theta(u_2, a, v)$

(ii) $\theta(u, a + b, v) = \theta(u, a, v) + \theta(u_2, b, v)$

(iii) $\theta(u, a, v_1 + v_2) = \theta(u, a, v_1) + \theta(u, a, v_2)$

1. If $\theta(u, a, v)$ and $\varphi(a, u, b)$ are represented by ternary ways as uav and aub respectively, then with these notations we must have

$$(uav)\ bw = ua(v\ bw) = u(av\ b)w$$

Then B is called a Gamma ring

2. Let V and Γ be two linear spaces over the field F. V is said to be a Gamma algebra over F, denoted by (V, Γ) , if for x, y, z $\in V$; $a, b \in \Gamma$; $\alpha \in F$, the following conditions are satisfied :
   (iv) $xay \in V$
   (v) $(xay)\, bz = xa(y\, bz)$
   (vi) $\alpha\, (x\gamma y) = (\alpha x)ay = x(\alpha a)y = xa(\alpha y)$
   (vii) xa $(y + z) = xay + xaz$; $x(a + b)y = xay + xby$
   and $(x + y)az = xaz + yaz$
   If V and Γ are normed linear spaces over F, then $\Gamma -$ algebra V is called a Gamma normed algebra if conditions (iv), (v), (vi), (vii) together with (viii)
   $\|xay\| \leq \|x\|_V\, \|a\|_\Gamma\, \|y\|_V$ hold.
   If $xay = 0$ for all x, y $\in V$; $a \in \Gamma$ implies $a = 0$, then the pair (V, Γ) is called a week $\boldsymbol{\Gamma_N} -$ algebra.
Gamma normed algebra (V, Γ) is called a Gamma Banach algebra if V is a Banach space.

Definition 2.2: Let (B,Γ) be a $\Gamma -$ Banach Algebra. Then a mapping $\varphi: B \rightarrow B$ is said to be a $\boldsymbol{\Gamma -}$ **derivation** if φ (u + v) = φ(u) + φ(v) and φ(uav) = φ(u)av + ua $\varphi(v)$, for all u,v ∈ B and a∈ Γ .

Definition 2.3: Let (B,Γ) be a $\Gamma -$ Banach Algebra. Then an additive mapping $\psi: B \rightarrow B$ is said to be a generalized Γ– derivation on (B, Γ) if $\psi(uav) = \psi(u)av + ua\varphi(v)$, where φ is a Γ- derivation on (B, Γ) .

Definition 2.4: Let (B,Γ) be a $\Gamma -$ Banach Algebra. Then an additive mapping $\psi: B \rightarrow B$ is said to be a generalized Jordan Γ– derivation on (B, Γ) if $\psi(uau) = \psi(u)au + ua\varphi(u)$, where φ is a Γ- derivation on (B,Γ).

Definition 2.5: Let (B,Γ) be a $\Gamma -$ Banach Algebra. Then a $\Gamma -$derivation $\varphi: B \rightarrow B$ is said to be inner on (B, Γ) if there exists an element w in B such that $\varphi(uau) = wau - uaw$. φ is said to be a generalized inner derivation if $\varphi(uau) = wau - uas$, where w and s are two fixed elements in B.

Definition 2.6: A map $\varphi: B \rightarrow B$ is said to be a $\Gamma -$homomorphism if $\varphi(uav) = \varphi(u)a\varphi(v)$ for all u, v ∈ B and a∈ Γ . The multiplicative centre Z of B is the set:

Z=$\{w \in B: wau = uaw \text{ } for\ all\ u \in B\ and\ a \in \Gamma\}$

Definition 2.7 : Let (B,Γ) be a $\Gamma -$ ring and $\Psi = (\psi_i)_{i\in N}$ be a family of additive mappings of B into itself with $\psi_0 = id.$ Then Ψ is said to be a generalized higher derivation on B if there exists a higher derivation $\Phi = (\varphi_i)_{i\in N}$ on B such that

$$\psi_n(uav) = \textstyle\sum_{i+j=n} \psi_i(u)a\varphi_j\,(v),\ \text{n}\in N$$

Ψ is said to be a Jordan generalized higher derivation on B if there exists Jordan higher derivations $\Phi = (\varphi_i)_{i\in N}$ on B such that

$$\psi_n(uau) = \textstyle\sum_{i+j=n} \psi_i(u)a\varphi_j\,(u),\ \text{n}\in N$$

Ψ is said to be a Jordan generalized triple higher derivation on B if there exists Jordan higher triple derivations $\Phi = (\varphi_i)_{i\in N}$ on B such that

$$\psi_n(uavbu) = \textstyle\sum_{i+j+l=n} \psi_i(u)a\varphi_j\,(v)b\varphi_l(u),\ \text{n}\in N$$

Definition 2.8: Let (B1, Γ 1) and (B2, Γ 2) be two gamma Banach algebras. Let

B = B1 × B2 and Γ = Γ 1 × Γ 2. Then we define addition and multiplication on B and Γ by, (u1, u2) + (v1, v2) = (u1 + v1, u2 + v2), (a1, a2) + (b1, b2) = (a1 + b1, a2 + b2) and
(u1, u2)(a1, a2)(v1, v2) = (u1a1v1, u2a2v2) for every (u1, u2), (v1, v2) ∈ B and (a1, a2), (b1, b2) ∈ ΓXΓ .
Again, if x = $\sum_i (u_i \otimes v_i)$ is an element of the algebraic tensor product B1$\otimes B_2$ then the projective norm p is defined by p(x) = inf{ $\sum_i \|u_i\|\, \|v_i\| : u_i \in X,\ v \in Y\}$ where the infimum is taken over all finite representations of x. Further the weak norm W on x is defined by

W(x) = sup$\{\left|\sum_i \xi_1\left(f(u_i)\right).\xi_2(g(v_i))\right| : f \in {B_1}^*, g \in {B_2}^*, \|f\| \leq 1, \|g\| \leq 1\}$

Here $B_1{}^*$ and $B_2{}^*$ are respective dual spaces of B1 and B2 . The projective tensor product $B_1 \otimes_p B_2$ and the weak tensor product $B_1 \otimes_w B_2$ are the completions of $B_1 \otimes B_2$ with their respective norms. For details, see Bonsall and Duncan's book., [1,2,3,18].

The projective tensor product (B1, Γ_1) $\otimes_p$ (B_2, Γ_2) with the projective norm is a $\Gamma_1 \otimes \Gamma_2$- Banach algebra over the field F, where multiplication is defined by the formula

$$(u \otimes v)(a \otimes b)(u' \otimes v') = (uau') \otimes (vbv')\text{ , where } u, v \in B_1; u', v' \in B_2;\ a \in \Gamma_1 ; b \in \Gamma_2 .$$

We now set forward to our main results.

3. Main Results:

3.1: Let (B1, Γ 1) and (B2, Γ 2) be two gamma Banach algebras, and (B, Γ) be their Projective tensor product. Then we get the following results:
(I) Every pair of Gamma- derivations φ1 and φ 2 on (B1, Γ 1) and (B2, Γ 2), respectively give rise to a Gamma-derivation φ on (B, Γ).

Proof: We define a mapping φ: B → B by φ (x) = φ ((u1, u2)) = (φ 1(u1), φ 2(u2)). Clearly, φ is a well defined mapping. We show that φ is a derivation on (B, Γ).

Let u = (u1, u2), v = (v1, v2) ∈ B and a = (a1, a2) ∈ Γ be any elements. Then

φ (u + v) = φ ((u1, u2) + (v1, v2)) = φ ((u1 + v1), (u2 + v2))

= (φ 1(u1 + v1), φ 2(u2 + v2))

= (φ 1(u1) + φ 1(v1), φ 2(u2) + φ 2(v2)), [Since φ 1 and φ 2 are additive mappings]

= (φ 1(u1), φ 2(u2)) + (φ 1(v1), φ 2(v2))

= φ ((u1, u2)) + φ ((v1, v2)) = φ (u) + φ (v)

Thus, φ (u + v) = φ (u) + φ (v) , ∀u, v ∈ B which implies that φ is additive. Again, φ (uav) = φ ((u1, u2)(a1, a2)(v1, v2)) = φ ((u1a1v1, u2a2v2))

= (φ 1(u1a1v1), φ 2(u2a2v2))

= (φ 1(u1)a1v1 + u1a1 φ 1(v1), φ 2(u2)a2 v 2 + u2a2 φ 2(v2)) [Since φ1 and φ2 are Gamma-derivations on (B1, Γ 1) and (B2, Γ 2) respectively.

= (φ 1(u1)a1v1, φ 2(u2)a2v2) + (u1a1 φ 1(v1), u2a22 φ (v2))

= (φ 1(u1), φ 2(u2))(a1, a2)(v1, v2) + (u1, u2)(a1, a2)(φ 1(u1), φ 2(v2))

= φ ((u1, u2))(a1, a2)(v1, v2) + (u1, u2)(a1, a2) φ ((v1, v2))

= φ (u)av + ua φ (v)

Thus, φ (uav) = φ (u)av + ua φ (y) ∀ u, v ∈ B and a ∈ Γ
So φ is a Gamma-derivation on (B, Γ) and hence the result.

(II) For every generalized Gamma-derivations f1 and f2 on (B1, Γ 1) and (B2,Γ 2) respectively give rise to a generalized Gamma-derivation f on (B, Γ) .
Proof: Let f1 be a generalisd derivation on (B1, Γ 1) with respect to the Gamma- derivation

φ 1: B1 → B1 and f2 be a generalized Gamma-derivation on (B2, Γ2)

with respect to the Gamma-derivation φ 2: B2 → B2.

We define the mappings f: B → B and d: B → B by

f(u) = f((u1, u2)) = (f1(u1), f2(u2)) and

φ (x) = φ ((u1, u2)) = (φ 1(u1), φ 2(u2)) for all u = (u1, u2) ∈ B .

Then obviously f is an additive mapping and φ is a Gamma-derivation on B.

We shall show that f is a generalized derivation on B with respect to the derivation φ on B.

Let u = (u1, u2), v = (v1, v2) ∈ B and a = (a1, a2) ∈ Γ be any elements. Then

f(uav) = f((u1, u2)(a1, a2)(v1, v2)) = f((u1a1v1, u2a2v2))

= (f1(u1a1v1), f2(u2a2v2))

= (f1(u1)a1v1 + u1a11 φ (v1), f2(u2)a2v2 + u2a2 φ 2(v2)) [Since f1 and f2 are generalized derivations on (B1, Γ 1) and (B2, Γ 2) respectively.]

= (f1(u1)a1v1, f2(u2)a2v2) + (u1a1 φ 1(v1), u2a22 φ (v2))

= (f1(u1), f2(u2))(a1, a2)(v1, v2) + (u1, u2)(a1, a2)(φ 1(v1), 2 φ (v2))

= f((u1, u2))(a1, a2)(v1, v2) + (u1, u2)(a1, a2) φ ((v1, v2))

= f(u)av + ua φ (y)

Thus, f(uav) = f(u)av + ua$d$(v) ∀ u, v ∈ B and a ∈ Γ

Hence f is a generalized Gamma-derivation on B with respect to the Gamma- derivation φ on B.

(III) Two inner Gamma-derivations φ 1 and φ 2 on (B1, Γ1) and (B2, Γ 2) respectively give rise to an inner Gamma-derivation φ on (B, Γ) .

Proof: Let φ 1 be an inner Gamma-derivation on (B1, Γ 1) with respect to the element u ∈ B1and φ2 be an inner Gamma-derivation on (B2, Γ 2) with respect to the element v ∈ B2. We define a mapping φ: B → B by φ (u) = φ ((u1, u2)) = (φ 1(u1), φ 2(u2)) ∀u = (u1, u2) ∈ B. Then, φ is well defined as well as additive. Let u = (u1, u2) ∈ B and a = (a1, a2) ∈ Γ be any two elements. Then

φ (uau) = φ ((u1, u2)(a1, a2)(u1, u2)) = φ ((u1a1u1, u2a2u2))

= (φ 1(u1a1u1), φ 2(u2a2u2))

= (αa1x1 − x1a1α, βa2u2 − u2a2β) [Since φ 1 and φ2 are inner derivations on (B1, Γ 1) and (B2, Γ 2) w.r.t. α and β respectively]

= (αa1u1, βa2u2) − (u1a1α, u2a2β)

= (α, β)(a1, a2)(u1, u2) − (u1, u2)(a1, a2)(α, β)

= $m$au − ua$m$ $where$ m = (α, β) ∈ B

Thus φ is an inner derivation on (B, Γ) with respect to the element m ∈ B.

(IV) Every two Jordan derivations J1 and J2 on (B1, Γ 1) and (B2, Γ 2) respectively give rise to a Jordan derivation J on (B, Γ) defined by J1 and J2.
Proof: We define a map J: B → B by J(u) = J((u1, u2)) = (J1(u1), J2(u2))

∀u = (u1, u2) ∈ B. Then J is an well defined as well as additive mapping.

Let u = (u1, u2) ∈ B and a = (a1, a2) ∈ Γ be any two elements. Then

J(uau) = J((u1, u2)(a1, a2)(u1, u2)) = J((u1a1u1, u2a2u2))

= (J1(u1a1u1), J2(u2a2u2))

= (J1(u1)a1u1 + u1a1J1(u1), J2(u2)a2u2 + u2a2J2(u2)) [Since J1 and J2 are Jordan derivations on (B1, Γ 1) and (B2, Γ 2) respectively]

= (J1(u1)a1u1, J2(u2)a2u2) + (u1a1J1(u1), u2a2J2(u2))

= (J1(u1), J2(u2))(a1, a2)(u1, u2) + (u1, u2)(a1, a2)(J1(u1), J2(u2))

= J((u1, u2))(a1, a2)(u1, u2) + (u1, u2)(a1, a2)J((u1, u2)) = J(u)au + ua$J$(u)

Thus, J(uau) = J(u)au + ua$J$(u) ∀ u ∈ X $and\ \alpha \in$ F

So J is a Jordan derivation on (B, □) defined by J1 and J2 ; and hence the result. Similarly we can show some other enlightening results highlighted below:

(V) Every two generalized Jordan derivations J1 and J2 on (B1, Γ 1) and (B2, Γ 2) respectively give rise to a generalized Jordan derivation J on (B, Γ) constructed with the help of J1 and J2.

(VI) Every two generalized inner derivations on (B1, Γ 1) and (B2, Γ 2) respectively give rise to a generalized inner derivation on (B, Γ).
Now we shall discuss the norm of a derivation.

4. THE NORM OF $\boldsymbol{\varphi}$

We now shift our attention to study the possibility of the result, $\|\boldsymbol{\varphi}\| = \|\boldsymbol{\varphi}_1\| + \|\boldsymbol{\varphi}_2\|$, when $\boldsymbol{\varphi}, \boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ are related as the above.

THEOREM 3.1. If $\boldsymbol{\varphi}, \boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ are related as the above, then

$$\|\boldsymbol{\varphi}\| \leq \|\boldsymbol{\varphi}_1\| + \|\boldsymbol{\varphi}_2\| \leq 2\|\boldsymbol{\varphi}\|.$$

PROOF. For each $u \in$ (B, Γ) $\otimes_p (B', \Gamma')$ with $\|u\|_p = 1$ and for each $\varepsilon > 0$, ∃ a (finite) representation $u = \sum_i x_i \otimes y_i$ such that $\|u\|_p + \varepsilon \geq \sum_i \|x_i\|\, \|y_i\|$.

$$\text{Now, } \|\boldsymbol{\varphi}\| = \sup_u \{\|\boldsymbol{\varphi} u\|_p \mid \|u\|_p = 1\}$$

$$= \sup_u \{\|\textstyle\sum_i [\boldsymbol{\varphi}_1 x_i \otimes y_i + x_i \otimes \boldsymbol{\varphi}_2 y_i]\|_p \mid \|u\|_p = 1\}$$

$$\leq \sup_u \{\textstyle\sum_i [\|\boldsymbol{\varphi}_1 x_i \otimes y_i\|_p + \|x_i \otimes D_2 y_i\|_p] \mid \|u\|_p = 1\}$$

$$= \sup_u \{\textstyle\sum_i [\|\boldsymbol{\varphi}_1 x_i\|\, \|y_i\| + \|x_i\|\, \|\boldsymbol{\varphi}_2 y_i\|] \mid \|u\|_p = 1\}$$

$$\leq \sup_u \{\textstyle\sum_i [\|\boldsymbol{\varphi}_1\|\, \|x_i\|\, \|y_i\| + \|x_i\|\, \|\boldsymbol{\varphi}_2\| \|y_i\|] \mid \|u\|_p = 1\}$$

$$\leq (\|\boldsymbol{\varphi}_1\| + \|\boldsymbol{\varphi}_2\|) \sup_u \{1 + \varepsilon \mid \|u\|_p = 1\}$$

$$= (\|\boldsymbol{\varphi}_1\| + \|\boldsymbol{\varphi}_2\|)(1 + \varepsilon)$$

Since, ε was arbitrary, it follows that $\|\boldsymbol{\varphi}\| \leq \|\boldsymbol{\varphi}_1\| + \|\boldsymbol{\varphi}_2\|$ …. (3.1)

Next, let $x \in B$ be such that $\|x\| = 1$. Then $\left\|{}^{x}/_{k_2} \otimes 1_{\alpha'}\right\| = \left\|{}^{x}/_{k_2}\right\| \|1_{\alpha'}\| = 1$

$$\text{Now, } \|\boldsymbol{\varphi}\| = \sup_u \{\|\boldsymbol{\varphi} u\|_p \mid \|u\|_p = 1\}$$

$$\geq \left\|\boldsymbol{\varphi}({}^{x}/_{k_2} \otimes 1_{\alpha'})\right\|_p = \left\|\boldsymbol{\varphi}_1({}^{x}/_{k_2}) \otimes 1_{\alpha'}\right\|_p, \text{ (since } \boldsymbol{\varphi}_2(1_{\alpha'}) = 0 = \|\boldsymbol{\varphi}_1 x\|$$

Thus, $\|\boldsymbol{\varphi}_1 x\| \leq \|\boldsymbol{\varphi}\|$ for every $x \in B$ with $\|x\| = 1$. This gives $\|\boldsymbol{\varphi}_1\| \leq \|\boldsymbol{\varphi}\|$

Similarly, we can prove that $\|\boldsymbol{\varphi}_2\| \leq \|\boldsymbol{\varphi}\|$. Hence, we have $\|\boldsymbol{\varphi}_1\| + \|\boldsymbol{\varphi}_2\| \leq 2\|\boldsymbol{\varphi}\|$ ….. (3.2)

The inequalities (3.1) and (3.2) together imply $\|\boldsymbol{\varphi}\| \leq \|\boldsymbol{\varphi}_1\| + \|\boldsymbol{\varphi}_2\| \leq 2\|\boldsymbol{\varphi}\|$.
which is the required result .
Problem : Can we extend all the above results to the Projective Product of n number Gamma Banach Algebras ?

References:

[1] D K Bhattacharjee and A K Maity, "Regular Representation of Gamma Banach Algebras" Journal of Pure Mathematics, Calcutta university, India, 12, 34-42, 2001.
[2] F. F. Bonsall and J. Duncan, "Complete Normed Algebra"Springer Verlag, 1973
[3] M. R. Hestenes, "A ternary algebra with applications to matrices and linear transformations", Arch. Rational Mech. Anal. 11, 138-194, 1962.
[4] N. Nobusawa, "Structure theorems of a moduleover a ring with bilinear mapping", Canad. Math. Bull., 10 649-652, 1967.
[5] W. E. Barnes, "On the Γ-rings of Nobusawa", Pacific J. Math, 18, 411-422, 1966.
[6] J. Luh, "On the theory of Simple Γ- rings", Michigan Math. J., 16, 65-75, 1969.
[7] J. Luh, "The structure of primitive gamma-rings", Osaka J. Math., 7, 267-274, 1970.
[8] W.G. Lister, "Ternary rings", Trans. Amer. Math. Soc., 154, 37-55, 1971.
[9] W.E. Coppage and J. Luh, "Radicals of gamma rings", J. Math. Soc. Japan, 23, 40-52, 1971.
[10] H.C Myung, "A characterization of the Jacobson radical in ternary algebras", Proc. Amer. Math. Soc., 38, 228-234, 1973.
[11] S. Kyuno, "On the radicals of Γ- rings", Osaka J. Math., 12, 639-645, 1975.
[12] S. Kyuno, "Prime ideals in gamma-rings", Pacific J.Math. 98, 375-379, 1982.
[13] T. S.Ravisankar and U. S.Shukla, "Structure of Γ- rings", Pacific J. Math., 80, 539-559, 1979.
[14] R. A. Stephenson, "Jacobson structure theory for Hestenes ternary rings", Trans. Amer. Math. Soc., 177, 91-98, 1973.
[15] G. L. Booth, "A Brown-McCoy radical for Γ- rings", preprint.
[16] M. Parvathi and P. A. Rajendran,"Gamma-rings and Morita equivalence", Comm. in Algebra, 12, 1781-1786, 1984.
[17] M. Parvathi and P. R. Adhikari, "Γ- rings and Morita equivalence for rings with local units", Comm. in Algebra, 13, 585-598, 1985.
[18] Ranu Paul, "On Gamma Derivation in the Projective Product of Gamma Rings "International Journal of Algebra, 9, 291-300, 2015.