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FEIGENBAUM UNIVERSALITY, TIME SERIES ANALYSIS AND LYAPUNOV EXPONENTS IN NONLINEAR TWO DIMENSIONAL CHAOTIC MODELS

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Abstract: In this paper, we consider a two dimensional nonlinear chaotic model as $\emptyset_\beta(x,y) = (\alpha x + y, \beta x + \gamma y - x^3)$ where α,β,γ are adjustable parameters. M.J. Feigenbaum showed around 1980 how a route can be established from a regular system to a chaotic system in many nonlinear systems. Here we establish the universal route with the above mentioned model by determining the sequence of bifurcation points with the help of numerical methods and computer software. Time series analysis is carried out with different graphs in order to reveal how stability and instability of the periodic points appear in different ranges of the parameters. We evaluate Lyapunov exponents along with their graphs in order to confirm the regular and chaotic regions of the system. Different techniques are applied how to control the chaos i.e., how to go from the chaotic region to the regular one. Many other relevant results are discussed, and a few open problems are posed.

Keywords: Feigenbaum Universality; Bifurcations; Time Series Analysis; Lyapunov Exponents.

1. Introduction:

In the 21st Century, there has been a remarkable increase of research interest in dynamical systems and fractal geometry. However, during the past century Feigenbaum has proposed [6] the period-doubling phenomena of one-dimensional maps of the form

$$x_{n+1} = f_m(x_n) = f(x_n, m)$$

The foundation of his theory was that, when an orbit is optimally stable, the arrangement of periodic points that make up it may be described by a set of universal functions. These are roughly the functions

$$(-1)^l \xi_l f^{2^l b}(x / \xi_l, \widehat{m_{l+n}})$$

With a sufficiently big l(l is a positive integer) and adhering to the functional recursion formula, where is the f^kmaximum stability point of an orbit, stands for $k^{th}$the iterate of f, ξ_k is a scaling factor and $m = \widehat{m_k}$ is the iterate of $2^k b -$ period orbit. Note, the points of maximum stability also converge to m in the limit in addition to the periodic orbit bifurcation points denoted by m_l for$2^l b$ -periodic orbit. It can be demonstrated that the convergence rate of $\widehat{m_l}$the so-called Feigenbaum ratio δ is determined as an eigenvalue associated with a linear functional equation that is derived from a fixed point equation for the recursion formula mentioned above. This leads to the important conclusion that the convergence rate is universal. The value of δ in the dissipative situation is 4.6692016091029….., and 8.721097200...in the conservative case.

Furthermore, the sequence $\{m_n\}$ is set up such that a periodic trajectory of period develops at 2^n and that all other periodic trajectories of period $2^r(r < n)$ continue to be unstable. In addition, the Feigenbaum theory claims that all neighbouring points, excluding those that are part of these unstable orbits and their stable manifolds, are attracted to F under iterations of the family f at $m = m_\infty$[8]. This invariant set F of Cantor type is encompassed by infinitely many unstable periodic orbits of period 2^n(n=0,1,2,…)[1,2,12,13].

2. Methodology:

The main aim in this paper is to demonstrate the above well-defined objectives of a two-dimensional discrete map:

$$\emptyset_\beta = (f, g): R^2 \to R^2 \;. \qquad (1)$$

defined by f(x,y)=αx+y, g(x,y)=βx+γy-x^3,where α,β and γare adjustable parameters. By creating some suitable numerical algorithms, we set out our journey for the sole study and investigation.

For our convenience we write $\alpha = \frac{1}{3}$ and $\gamma = -1$.The Jacobian of ourfunction i

$$J_1 = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 1 \\ \beta - 3x^2 & -1 \end{bmatrix}$$

And det $(J_1) = 2\beta - \frac{13}{3}$. Here, φ_β is dissipative if|β|<$\frac{13}{3}$, area-preserving if β=$\frac{13}{3}$and area-expanding if|β|>$\frac{13}{3}$. We want to emphasize that the stability theory and our map's Jacobian matrix are closely related. If $\lambda_1, \lambda_2$are the eigenvalues of J_1, then as is widely known,

$$\lambda_{1,2} = \frac{1}{3}\left[(trace(J_1) + \frac{1}{3}) \pm \sqrt{D}\right] \text{, where, as usual, we define}$$

$$trace(J_1) = f_x + g_y, det(J_1) = f_x g_y - f_y g_{x,}$$

$$D = [1 - 9det(J_1)],\ \lambda_1 + \lambda_2 = trace(J_1) and\ \lambda_1\lambda_2 = det(J_1).$$

These relationships provide us with $\lambda_1 + \lambda_2 = -\frac{2}{3}$and $\lambda_1\lambda_2 = \frac{18\beta - 39}{9}$.
Now, one of the eigenvalues of J_1 has to be -1 for a period-doubling bifurcation. As a result, the previous eigenvalue relations yield the equation $\lambda_1 - 1 = -\frac{2}{3}$, and when we plug x=$\sqrt{\beta - \frac{4}{3}}$ this equation into another, we obtain the first period-doubling bifurcation point as β=2 . The fixed point remains stable for values of β falling within the interval I=(0,2) and a stable periodic trajectory with period 1 emerges around it . The stable fixed point gradually becomes unstable as the value of β is increased, and two points, such as $\overline{x_{21}}(\beta)$and$\overline{x_{22}}(\beta)$, create a stable periodic trajectory with a period of 2 around it.

We must now redirect our focus from the first iteration of our map to the second iteration, which is provided by ϕ^2_β.

$$\emptyset^2_\beta(x, y) = \left(\frac{1}{9}x + \beta x - \frac{2}{3}y - x^3, \beta y - \frac{2}{3}\beta x + y + \frac{26}{27}x^3 - y^3 - \frac{1}{3}x^2 y - xy^2\right)$$

Solving the equation φ^2_β (x,y)= (x,y) yields the periodic points of period-2 for the map, which are the fixed points of ϕ^2_β .The solutions of this 9th degree equation are found to be

(x=0,y=0),$\left(x = \sqrt{\beta}, y = \frac{4\sqrt{\beta}}{3}\right)$,$\left(x = \sqrt{\beta}, y = \frac{-4\sqrt{\beta}}{3}\right)$,$\left(x = -\frac{\sqrt{-4+3\beta}}{\sqrt{3}}, y = -\frac{2\sqrt{-4+3\beta}}{3\sqrt{3}}\right)$,

$\left(x = \frac{\sqrt{-4+3\beta}}{\sqrt{3}}, y = \frac{2\sqrt{-4+3\beta}}{3\sqrt{3}}\right)$,$\Bigg(x = -\frac{\sqrt{-2+\sqrt{3\beta}-\sqrt{3}\sqrt{-4-4\beta+3\beta^2}}}{\sqrt{6}}, y =$
$\frac{1}{6}\Big\{4\sqrt{\frac{2}{3}}\sqrt{-2 + 3\beta - \sqrt{3}\sqrt{-4 - 4\beta + 3\beta^2}} - 3\sqrt{\frac{3}{2}}\beta\sqrt{-2 + 3\beta - \sqrt{3}\sqrt{-4 - 4\beta + 3\beta^2}} + \frac{1}{2}\sqrt{\frac{3}{2}}\big(-2 + 3\beta -$
$\sqrt{3}\sqrt{-4 - 4\beta + 3\beta^2}\big)^{3/2}\Big\}\Bigg)$, $\Bigg(x = \frac{\sqrt{-2+\sqrt{3\beta}-\sqrt{3}\sqrt{-4-4\beta+3\beta^2}}}{\sqrt{6}}, y =$

$\frac{1}{6}\left\{-4\sqrt{\frac{2}{3}}\sqrt{-2+3\beta-\sqrt{3}\sqrt{-4-4\beta+3\beta^2}}+3\sqrt{\frac{3}{2}}\beta\sqrt{-2+3\beta-\sqrt{3}\sqrt{-4-4\beta+3\beta^2}}-\frac{1}{2}\sqrt{\frac{3}{2}}\left(-2+3\beta-\sqrt{3}\sqrt{-4-4\beta+3\beta^2}\right)^{3/2}\right\}\Bigg)$, $\Bigg(x=-\frac{\sqrt{-2+\sqrt{3\beta}+\sqrt{3}\sqrt{-4-4\beta+3\beta^2}}}{\sqrt{6}}, y=\frac{1}{6}\left\{4\sqrt{\frac{2}{3}}\sqrt{-2+3\beta+\sqrt{3}\sqrt{-4-4\beta+3\beta^2}}-3\sqrt{\frac{3}{2}}\beta\sqrt{-2+3\beta+\sqrt{3}\sqrt{-4-4\beta+3\beta^2}}+\frac{1}{2}\sqrt{\frac{3}{2}}\left(-2+3\beta+\sqrt{3}\sqrt{-4-4\beta+3\beta^2}\right)^{3/2}\right\}\Bigg)$, $\Bigg(x=\frac{\sqrt{-2+\sqrt{3\beta}+\sqrt{3}\sqrt{-4-4\beta+3\beta^2}}}{\sqrt{6}}, y=\frac{1}{6}\left\{-4\sqrt{\frac{2}{3}}\sqrt{-2+3\beta+\sqrt{3}\sqrt{-4-4\beta+3\beta^2}}+3\sqrt{\frac{3}{2}}\beta\sqrt{-2+3\beta+\sqrt{3}\sqrt{-4-4\beta+3\beta^2}}-\frac{1}{2}\sqrt{\frac{3}{2}}\left(-2+3\beta+\sqrt{3}\sqrt{-4-4\beta+3\beta^2}\right)^{3/2}\right\}\Bigg)$.

The Jacobian matrix J2 (say) of the second iteration of our map is given by

$$J_2=\begin{bmatrix} \frac{1}{9}-3x^2+\beta & -\frac{2}{3} \\ 3x^2-\left(\frac{x}{3}+y\right)^2-\frac{2\beta}{3} & 1-3\left(\frac{x}{3}+y\right)^2+\beta \end{bmatrix}$$

If μ_1, μ_2 are the eigenvalues of J2, then $\mu_1+\mu_2=\frac{10}{9}-3\left[x^2+\left(\frac{x}{3}+y\right)^2\right]+2\beta$ and

$$\mu_1.\mu_2=\frac{1}{9}-\frac{10}{9}x^2+x^4-\frac{2xy}{3}+6x^3y-y^2+9x^2y^2+\frac{2}{3}\beta-\frac{10}{3}x^2\beta-2xy\beta-3y^2\beta+\beta^2$$

If we set $\mu_1=-1$, then the above equations lead to

$$x^4+\beta^2+\beta\left(\frac{2}{3}-\frac{10}{3}x^2-3y^2-2xy\right)+6x^3y+x^2\left(9y^2-\frac{40}{9}\right)-4y\left(y+\frac{2}{3}x\right)-\frac{20}{9}=0 \quad (2)$$

Putting $x=\frac{\sqrt{-4+3\beta}}{\sqrt{3}}$ and $y=\frac{2\sqrt{-4+3\beta}}{3\sqrt{3}}$ in (2) and solving for β, we get β=2.66666667 as the second bifurcation value. We must take into account the fourth iteration of our model after determining the second bifurcation value. In that situation, we have seen that the analytical discussion in the case of the periodic points of period-2 presents a tremendous computing challenge. Therefore, further analysis is essentially difficult at this point, and we must turn to a numerical method [18,20]. The numerical methods listed below are effective for our objective.

2.1 Numerical Method for Obtaining Periodic Point:

We have discovered that one of the most accurate numerical techniques for our needs is the Newton-Raphson formula (also known as Newton's iteration formula), which may be used to identify a periodic fixed point[9,12]. Furthermore, it provides a periodic point with quick convergence.
The Newton's iteration formula is

$$\overline{x_{n+1}}= D(\psi)(\overline{x_n})^{-1}\psi(\overline{x_n}), \qquad n=0,1,2,...$$

where$D(\psi)(\overline{x})$ is the Jacobian of the map ψ at the vector$\bar{x}$. We see that in our situation, where k is the suitable duration, this mapψ is equal toϕ_β^k-I. The zero(s) of a map are actually provided by the Newton formula, and in order to use this numerical tool with our map, one requires a number of the recurrence equations that are provided below. Let the initial data be (x0,y0), then

$$\emptyset_\beta(x_0,y_0)=(\alpha x_0+y_0, \beta x_0+\gamma y_0-x_0^3)=(x_1,y_1)$$

$$\emptyset_\beta^2(x_0,y_0)=\emptyset_\beta\left(\emptyset_\beta(x_0,y_0)\right)=\emptyset_\beta(x_1,y_1)=(x_2,y_2)$$

Preceding in this manner the following recurrence formula for the map $\phi_\beta(x,y)$ can be established as x_n=$\alpha x_{n-1}+y_{n-1}$ and $y_n=\beta x_{n-1}+\gamma y_{n-1}-x_{n-1}^3$; n=1,2,3,…

Since the Jacobian of ϕ_β^k(ktimes of the mapϕ_β) is the product of the Jacobian of each iteration of the map, we proceed as follows to describe our recurrence mechanism for the Jacobian Matrix. The Jacobian J1 for the transformation, $\emptyset_\beta(x_0,y_0)$=$(\alpha x_0+y_0,\beta x_0+\gamma y_0-x_0^3)$ is

$$J_1=\begin{bmatrix}\frac{1}{3} & 1\\ \beta-3x_0^2 & -1\end{bmatrix}=\begin{bmatrix}E_1 & F_1\\ G_1 & H_1\end{bmatrix},\text{ where } E_1=\tfrac{1}{3}, F_1=1, G_1=\beta-3x_0^2, H_1=-1.$$

Next the Jacobian J2 for the transformation $\phi_\beta^2=(x_0,y_0)=(x_2,y_2)$ is the product of the Jacobian for the transformation

$$\emptyset_\beta(x_1,y_1)=(\alpha x_1+y_1,\beta x_1+\gamma y_1-x_1^3)$$

And $\emptyset_\beta(x_0,y_0)$=$(\alpha x_0+y_0,\beta x_0+\gamma y_0-x_0^3)$

So, we obtain

$$J_2=\begin{bmatrix}\frac{1}{3} & 1\\ \beta-3x^2 & -1\end{bmatrix}\begin{bmatrix}E_1 & F_1\\ G_1 & H_1\end{bmatrix}=\begin{bmatrix}\frac{1}{3}E_1+G_1 & \frac{1}{3}F_1+H_1\\ (\beta-3x^2)E_1-G_1 & (\beta-3x^2)F_1-H_1\end{bmatrix}=\begin{bmatrix}E_2 & F_2\\ G_2 & H_2\end{bmatrix}$$

Where $E_2=\frac{1}{3}E_1+G_1$, $F_2=\frac{1}{3}F_1+H_1$, $G_2=(\beta-3x^2)E_1-G_1$, $H_2=(\beta-3x^2)F_1-H_1$.

Continuing in this manner, we obtain the Jacobian for ϕ_β^k as $J_m=\begin{bmatrix}E_m & F_m\\ G_m & H_m\end{bmatrix}$with a set of recursive formula as

$E_m=\frac{1}{3}E_{m-1}+G_{m-1}$, $F_m=\alpha F_{m-1}+H_{m-1}$, $G_m=(\beta-3x_{m-1}^2)E_{m-1}-G_{m-1}$, $H_m=(\beta-3x_{m-1}^2)F_{m-1}-H_{m-1}$where m= 2,3,4,…

Since the fixed point of the map ϕ_β is a zero of the map $\psi(x,y)=\phi_{\alpha\beta}(x,y)-(x,y)$ the Jacobian of ψ^k is given by $J_k-I=\begin{bmatrix}E_k-1 & F_k\\ G_k & H_k-1\end{bmatrix}$. Its inverse is

$$(J_k-I)^{-1}=\frac{1}{\Delta}\begin{bmatrix}H_k-1 & -F_k\\ -G_k & E_k-1\end{bmatrix}$$

where Δ= $(E_k-1)(H_k-1)F_kG_K$, the Jacobian determinant. So, Newton's method gives the following recurrence formula in order to yields a periodic point of ϕ_β^k

$$x_{n+1}=x_n-\frac{(H_k-1)({x^*}_n-x_n)-F_k(y_n^*-y_n)}{\Delta}$$

$$y_{n+1}=y_n-\frac{(-G_k)(x_n^*-x_n)+(E_k-1)(y_n^*-y_n)}{\Delta}$$

where $\emptyset^k(\overline{x_n})=\emptyset^k(x_n,y_n)=(x_n^*,y_n^*)$

2.2 Numerical Method for Finding Bifurcation Values:

The Jacobian Matrix of the map ϕ_β^k represented by Newton's technique first requires us to recall our recurrence relations, and the eigenvalue theory [10] which provide the relation at the bifurcation value $E_k+H_k=-1-\Delta$. Yet again, according to the Feigenbaum theory

$$\beta_{n+2} \approx \beta_{n+1} + \frac{\beta_{n+1}-\beta_n}{\delta} \quad (3)$$

where$n = 1,2,3, ...$, is the kth bifurcation point of the parameter β, and δ is the Feigenbaum universal constant.The first two bifurcation values,β_1 andβ_2, for our map have already been determined. Furthermore, with the value of α=0.3, it is simple to locate the periodic sites for these $\beta_1$andβ_2. We observe that if we put I=$E_k + H_k + 1 + \Delta$, I proves to be a function of the parameter β. When I(β) equals zero, the bifurcation value of β for the period k occurs. This indicates that one needs the zero of the function I(β), which is provided by the Secant Method in Numerical analysis[8,9,11], in order to obtain a bifurcation value of period k.

$$\beta_{n+1} = \beta_n - \frac{I(\beta_n)(\beta_n - \beta_{n-1})}{I(\beta_n) - I(\beta_{n-1})}$$

Then, an approximation of $\beta_3$is obtainedby using the relation (3). We use $\acute{\beta}_3$ and a slightly bigger value, say, $\acute{\beta}_3 + 10^{-4}$ as the two beginning values to apply this approach and finally achieve $\beta_3$as the Secant method requires two initial values. Similar to this, the same method is used to get the successive bifurcation values $\beta_4, \beta_5, ...$etc., according to our needs. We create the table below for the first ten bifurcation points for the parameter α= 1/3with the help of the C programming [6].

TABLE 1: The Period-Doubling Cascade

Period	Bifurcation Point	Value of δ	Corresponding Periodic point
1	2.00000000		(x=0.81696581, y=0.544331054)
2	2.66666667		(x=1.15470054, y= 0.76980036)
4	2.80944626	4.66932113	(x=1.21495388, y= 0.809969254)
8	2.84002528	4.66908…	(x=1.22747381, y= 0.818315874)
16	2.84657437	4.669201…	(x=1.23013862, y= 0.820092414)
32	2.84797698	4.66921…	(x=1.2307086, y= 0.8204724)
64	2.84827739	4.66898…	(x=1.230983064, y=0.82055376)
128	2.84834173	4.66910…	(x=1.2308567, y=0.82057118)
256	2.84835551	4.66908…	(x=1.23086237, y=0.820574914)
512	2.84835846	4.66912…	(x=1.23027826, y=0.820185507)

The following formula is used to compute the ratios of the bifurcation points' sequential separations:
In our situation, we compute δ as follows:

$$\delta_1 = \frac{\beta_2-\beta_1}{\beta_3-\beta_2} = 4.66932113, \delta_2 = \frac{\beta_3-\beta_2}{\beta_4-\beta_3} = 4.6690803, \delta_3 = \frac{\beta_4-\beta_3}{\beta_5-\beta_4} = 4.66920137$$
$$\delta_4 = \frac{\beta_5-\beta_4}{\beta_6-\beta_5} = 4.66921667, \delta_5 = \frac{\beta_6-\beta_5}{\beta_7-\beta_6} = 4.66898572, \delta_6 = \frac{\beta_7-\beta_6}{\beta_8-\beta_7} = 4.6691016$$
$$\delta_7 = \frac{\beta_8-\beta_7}{\beta_9-\beta_8} = 4.66908563, \delta_8 = \frac{\beta_9-\beta_8}{\beta_{10}-\beta_9} = 4.66911864 \ldots\ldots, etc.$$

and have a particular scaling associated with them. As k approaches infinity, the ratios tend to a constant; more specifically,

$$\lim_{k\to\infty}\left[\frac{\beta_k - \beta_{k-1}}{\beta_{k+1} - \beta_k}\right] = \delta = 4.66911864 \ldots$$

The 'universal' Feigenbaum constant δ = 4.66911864... is likewise found in this two-dimensional system.

3. Results and discussions

3.1: Time Series Analysis:

The term “time series” refers to a chronological list of observations made regarding a single variable. The observations are often made on a regular basis [3,10]. Two steps make up a time series analysis: (i) creating a model to represent the time series, and (ii) utilizing the model to anticipate (predict) future values. We have displayed the discrete time series below for the values of x_n and y_n to demonstrate the existence of periodic orbits and chaos, by using equation (1.1) as our model.

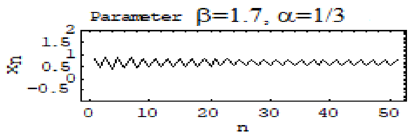


Fig 1 Showing period-1 behavior

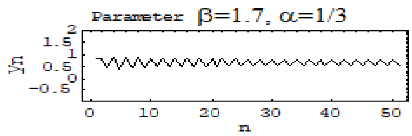


Fig 2 Showing period-1 behavior

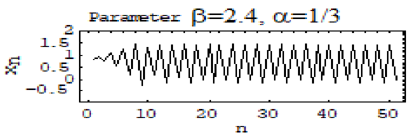


Fig 3 Showing period-2 behaviour

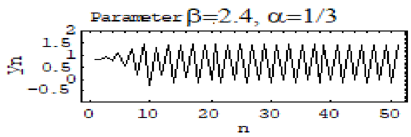


Fig 4 Showing period-2 behaviour

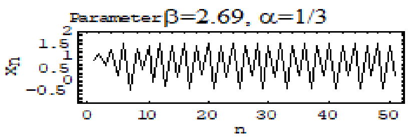


Fig 5 Showing period-4 behaviour

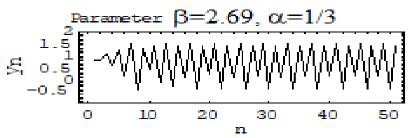


Fig 6 Showing period-4 behaviour

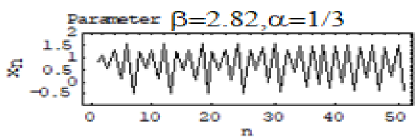


Fig 7 Showing period-8 behaviour

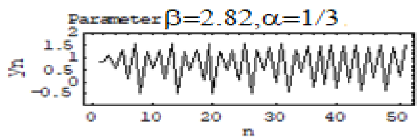


Fig 8 Showing period-8 behaviour

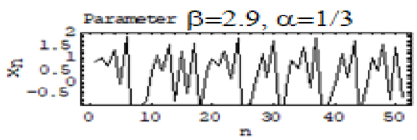


Fig 9 Showing chaotic behaviour

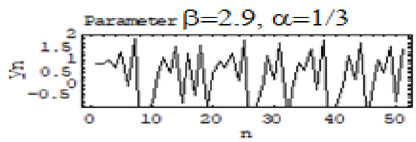


Fig 10 Showing chaotic behaviour

Our time-series graphs thus indicate the occurrence of many periodic orbits that eventually result in chaos.

3.2: Lyapunov Exponents:

The exponential divergence of originally adjacent trajectories is the key characteristic of chaotic dynamical systems, which is also known as their sensitive dependency on the initial conditions [16,17]. Divergence and convergence are present in different directions and so we need a collection of Lyapunov exponents to completely characterize an attractor's divergence and convergence behaviors. For various initial separation vector orientations, the rate of separation can vary. Since there are as many Lyapunov exponents as there are dimensions in the phase space, there is a wide range of them. The Maximal Lyapunov exponent (MLE), which defines how predictable a dynamical system is, is the one that is frequently mentioned because it is the greatest. A positive

MLE is typically seen as evidence of chaos in the system. The effect of the other exponents will gradually be eliminated over time due to the exponential growth rate, therefore take into account that any initial separation vector will normally contain some component in the direction of the MLE. The range of Lyapunov exponents for a dynamical system relies on the beginning point[11,12,19]. The Jacobian matrix is used to define the Lyapunov exponents, which explain the behavior of vectors in the tangent space of the phase space.

3.3: Lyapunov Exponents for Two-Dimensional Maps:

Consider a discrete dynamical system in two dimensions:

$$\bar{\mathbf{x}}_{n+1} = f(\bar{\mathbf{x}}_n),\ \bar{\mathbf{x}} \in \Re^{\mathbf{2}}$$

Let us examine the dynamics of the difference between the two paths $\overline{x_n}$ and$\overline{y_n} = \overline{x_n}$ +δ . Since we take into account tiny changes over the long run, the linearization of the map f, or its Jacobians, governs how these differences behave.

$$J(\bar{\mathbf{x}}_n) = \left(\frac{\partial f}{\partial \bar{\mathbf{x}}}\right)_{\bar{\mathbf{x}}=\bar{\mathbf{x}}_n}, or\ J_{ij}(\bar{\mathbf{x}}_n) = \left(\frac{\partial f_i}{\partial x_j}\right)_{\bar{\mathbf{x}}=\bar{\mathbf{x}}_n}$$

This produces a linear dynamical system for the perturbation δ with time-dependent coefficients $\boldsymbol{\delta}_{n+1} = J(\bar{\mathbf{x}}_n)\boldsymbol{\delta}_n$.

The long term dynamics is governed by the eigenvalues η_i of the product of the Jacobians

$$\prod_{n=1}^{N} J(\bar{\mathbf{x}}_n)\mathbf{v}_i^{(N)} = \boldsymbol{\eta}_i^{(N)}\mathbf{v}_i^{(N)}$$

with $\mathrm{v}_i(N)$ denoting the eigenvectors of the product of the N Jacobians. The Lyapunov exponent λ_i is then defined as the normalized logarithm of the modulus of the ith eigenvalue η_i of the product of all Jacobians along trajectory (in time order) in the limit of an infinitely long trajectory:

$$\lambda_i = \lim_{N\to\infty} \frac{1}{N} \log\left|\boldsymbol{\eta}_i^{(N)}\right|$$

Often, the eigenvalues are listed from largest to smallest in order of magnitude. However, in thecase of one-dimensional maps, the definition reduces to

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \log\left|f'(x_n)\right|$$

and the existence and uniqueness is established by the usual (Birkhoff) ergodic theorem.

3.4: Lyapunov Exponents for Our Map$\emptyset_{\boldsymbol{\beta}}$.

The eigenvalues in the limit of the following expression are used to calculate the Lyapunov exponent: $\sqrt[N]{G_0.G_1.G_2.....G_N}$

where N approaches infinity and G_k is the Jacobian of ϕ_β at the iterated point(x_k, y_k). In order to get close enough to the fixed points for the evaluation of Lyapunov exponents, we started with a point and repeated it, say, two thousand times. The eigenvalues of the resulting matrix M are then determined by finding $M = (G_0, G_1, G_2, \dots, G_N)$, where N=5000 (say). Then

$\lambda = \log\left[\frac{Eigenvalueof\ M}{N}\right]$ is the Lyapunov exponent.

The graph of Lyapunov exponents relative to parameter values from 1.5 to 2.85 is shown in figure 11. We have iterated from the starting point (1.44, 0.348) to evaluate the Lyapunov exponent with 5000 iterations. The curve of the Lyapunov exponent hits the horizontal line at the bifurcation points. The positive values indicate the irregular behavior of the system. We compose a computer program to calculate the Lyapunov exponents for the parameter values ranging from 1.5 to 2.96 for the graph of figure (12). It is interesting to note that at the accumulation point, the period-doubling regime terminates and the chaotic regime takes over. The existence of chaos in our model is supported by the positive Lyapunov values.

TABLE 2:

Control Parameter	Lyaponov exponent	Control Parameter	Lyaponov exponent
… … …	… … …	… … …	… … …
1.5	-0.000081	2.856327	0.003085
1.6145	-0.000484	2.878513	0.000192
1.78921	-0.000644	2.885142	0.007093
… … …	… … …	… … …	… … …
2.4561	-0.001222	2.94361	0.006507
2.61619	-0.000129	2.954476	0.003185
2.82579	-0.020619	2.96500	0.011006

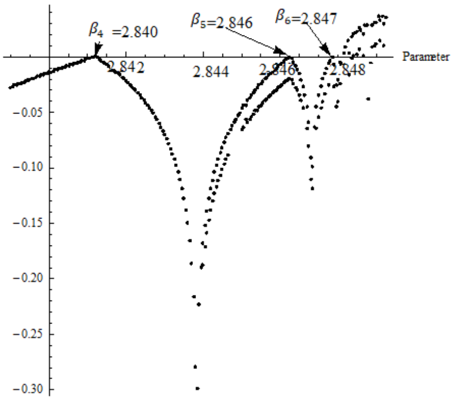


Fig 11 Graph of Lyapunov exponent versus control parameter.

Negative values indicate the existence of chaos. There are many practical reasons for controlling chaos. First of all, chaotic system response with little meaningful information content is unlikely to be useful. Secondly, chaos can lead systems to harmful situations, therefore chaos should be reduced as much as possible. Chaos is observed as undesirable part in engineering control practice. So, controlling of chaos is an essential part of study of chaos. The idea of "controlling chaos" was first suggested in a famous paper by Ott, Grebogi and Yorke in1995 and known as OGY method[12].After that many techniques for controlling chaos have been proposed in these decades. The proportional pulse method was introduced by Matias and Guemez[11]. After that N.P.Chau discussed in a similar manner but gave some restrictions on the initial conditions by which chaos can be controlled. We have taken the method of periodic proportional pulse and OGY method to control chaos.

3.5: Controlling of Chaos in Two Dimensional Map$\emptyset_{\boldsymbol{\beta}}$:

In this section, a two dimensional discrete map is considered. The map is

$$x_{n+1} = \alpha x_n + y_n \,;\; y_{n+1} = \beta x_n + \gamma y_n - x_n^3 \qquad (4)$$

whereβ is a control parameter. We have seen that the accumulation point of the system is 2.8302283462700...from where chaotic region starts. We consider the parameter value $\beta = \beta_0 = 2.83$ (say) which is far behind the accumulation point and shows a chaotic attractor [13,14,15]. Following time series plot beyond accumulation point (β=2.83) shows chaotic behavior of the model.

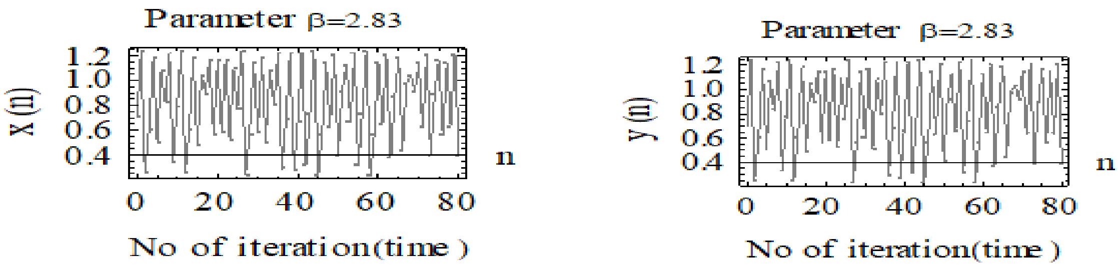


Fig 12: Time series plot showing irregular behavior at the control parameter $\beta = 2.83$.Abscissa represents the number of iterations, while the ordinate represents functional value at every iteration.

3.6: Control procedure for two dimensional map $\emptyset_{\beta}$:

Here we have considered the extended version of the above method (1).The procedure is as follows: Let us consider a two dimensional discrete system

$x_{n+1} = f(x_n, y_n)$, $y_{n+1} = g(x_n, y_n)$.The model can be written as

$X_{n+1} = F(X_n)$(5)

where X is a vector in R2 . To control the dynamics, kick is applied to the orbit of the composite map F^m , once every p steps, by multiplying the x component of the dynamics by a factor k_1 and the y component by a factor k_2 [4,5].Now the kicked map is defined a follows

$$H = KF^m, \tag{6}$$

where K is a diagonal matrix whose diagonal elements are k_1 and k_2 and F^mrepresents composition of F, m times. Any fixed point of H let's say X is stable if

$$H = KF^m(X) = X, (7)$$

and Jacobian matrix has two eigenvalues whose modulus <1 (unity). (8)
Now we have to determine the values of k_1 and k_2 such that chaos is controlled.
Proposed map and control procedure
The above extended version of the control procedure, is now applied in the two dimensional map $\emptyset_{\beta}$ withβ=2.8.

From (4) it is clear that lies β=2.83 in the chaotic region. The Jacobian matrix of (4) is $\begin{pmatrix} 0 & \frac{2}{3} \\ \beta - 3x^2 + 1 & 0 \end{pmatrix}$

So Jacobian of H will be $\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}\begin{pmatrix} 0 & \frac{2}{3} \\ \beta - 3x^2 + 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & k_1\frac{2}{3} \\ k_2(\beta - 3x^2 + 1) & 0 \end{pmatrix}$

The characteristic polynomial of the Jacobian matrix is given as follows:
$\lambda^2 - \lambda X + Y = 0$, where X= Sum of the diagonal elements= 0
and Y =$-\ \{k_1\frac{2}{3}\}\ \{k_2(\beta - 3x^2 + 1)\}$
The eigenvalues of the Jacobian matrix for any point (x, y) are given by

$$\lambda = \frac{X \pm \sqrt{X^2 - 4Y}}{2} \ => \lambda = \pm\sqrt{\{k_1\frac{2}{3}\}\ \{k_2(\beta - 3x^2 + 1)}.$$

The fixed point will be stable if $-1 < \lambda < 1$. (9)

We put $k_1 = \frac{x}{(1+\beta)y-\beta y^3}$ and $k_2 = \frac{y}{(1+\beta)x-\beta x^3}$. (10)

Now we pick those values of x, y which satisfy (13.1) and (13.2). For this purpose we make a suitable c-programming and draw the basin of attraction of period 1 i.e. for m=1, as shown in figure 13.

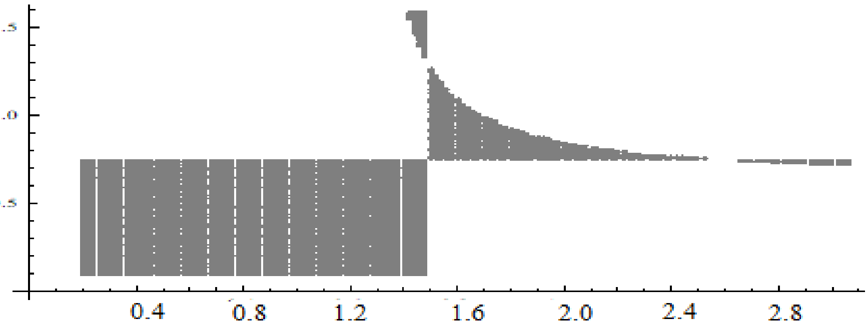


Fig 13.1a: Basin of attraction for period 1, where the points (x, y) satisfy equation (9) &(10).

Now any one point from the shaded portion is picked say (x=0.250000,y=0.500000) and determined kicking factor $k_1 = 0.235294, k_2 = 0.831169$ from(9) such that equation (8) is satisfied and eigenvalues $\lambda_1 = 0.772424$ and λ_2 ==-0.772424 are obtained. Applying control procedure with above kicking factor we have obtained following figur13.1b.

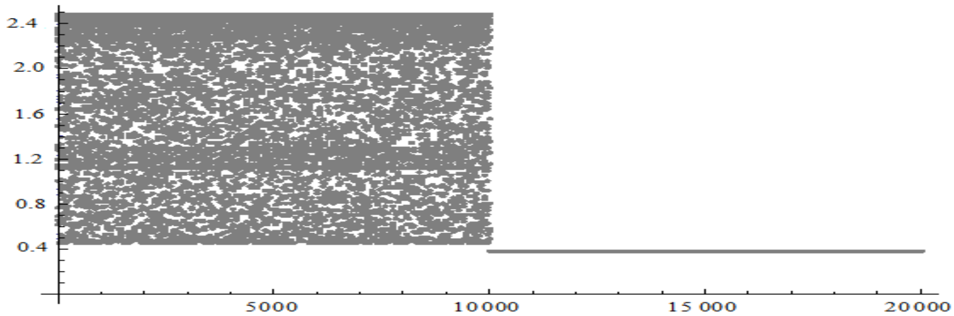


Fig13.1b: Chaos is controlled by taking initial value of (x, y) from the shaded portion of the fig13.1a.

In figure13.1b upto 10000 iterations are done at the parameter β=2.83 showing chaotic region and after that controlling parameters are switched on to get the periodic orbit of period one. Now for the other values of m, we have,

$$k_1 = \frac{x}{(1+r)y_{m-1} - r{y_{m-1}}^3}$$

and $k_2 = \frac{y}{(1+r)x_{m-1} - r{x_{m-1}}^3}$

where x_{m-1} is the first component of f^{m-1} and y_{m-1} is the second component of f^{m-1}.

Also the Jacobian matrix is given as $\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} \frac{\partial x_m}{\partial x} & \frac{\partial x_m}{\partial y} \\ \frac{\partial y_m}{\partial x} & \frac{\partial y_m}{\partial y} \end{pmatrix}$

Following the above discussed procedure we show controlling of chaos for different periodic orbit i.e. using different values of m.

Now for m=2: Basin of attraction is as shown in figure 13.2a.

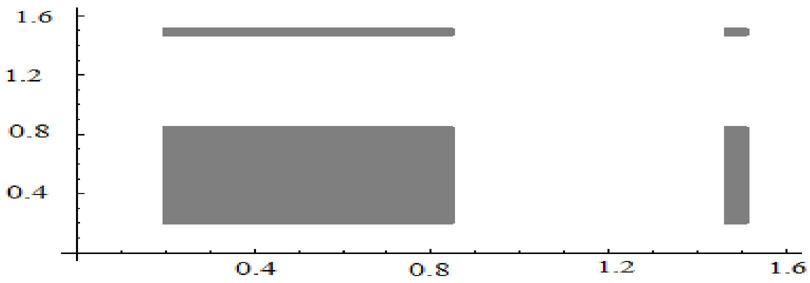
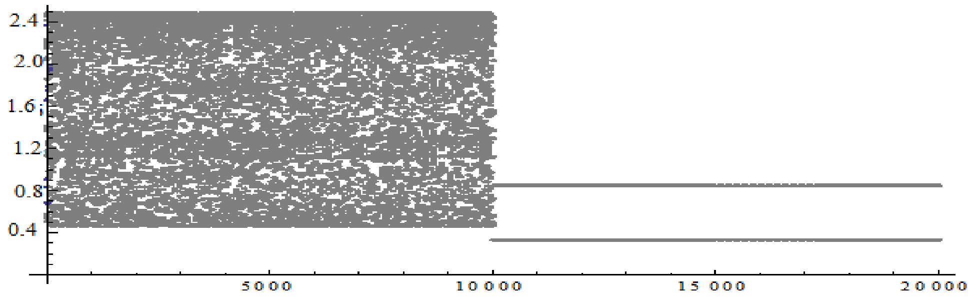


Fig13.2a: Basin of attraction for period -2. Abscissa is the x co-ordinate and ordinate is the y co-ordinate of (x, y).

Now shaded portion of the above figure represents the point (x, y) which may be stable fixed point by taking suitable values $k_1$and k_2.Now picking any one value of co-ordinate of (x, y) =(x=0.390000, y=0.120000) from the shaded portion we have obtained $k_1 = 0.332845, \mathrm{k}_2 = 0.170440$ and corresponding eigenvalues are $\lambda_1 = 0.872433$ and $\lambda_2 = -0.624028$.



Fi13.2b: Chaos is controlled by taking initial value of (x, y) from the shaded portion of the figure.

This way, we can show how the chaos can be controlled for other parameter values and periodic points. Feigenbaum Universality, Bifurcations, Time SeriesAnalysis, Lyapunov Exponents.

4. Conclusion:

Establishment of a mathematical link of regular behavior in a nonlinear system with its chaotic behavior is indeed a challenging research in this field. There are many nonlinear systems in which this kind of phenomenon cannot be established. Here we have framed a model with an affirmative answer. Different suitable numerical techniques and computer software programs are developed so as to obtain bifurcation values, the accumulation point dividing the regular region and the chaotic region, and finally, the Feigenbaum universal constant.

To confirm our results, Time series analysis and Lyapunov exponents are strong complimentary. Different analysis on the controlling of chaos is effectively highlighted. Whether these techniques can be applicable to higher dimensional systems is an interesting open problem. Our discussion and results have formed a strong foundation for studying the dynamical properties of a nonlinear system with fruitful outcomes.

Open Problems: (i) Can we apply our techniques for higher dimensional discrete systems?
(ii) Can we study this model for super critical and subcritical Hopf Bifurcations?

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