COMMON FIXED POINT THEOREMS IN COMPLETE PARTIAL METRIC SPACE

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Abstract: The objective of this article is to obtain coincidence points and common fixed point theorem in complete partial metric space. We provide some examples to support the usability of our results.

Keywords: Common fixed point; coincidence point; partial metric space

PACS: 47H10, 54H25

1. Introduction:

In spite of its simplicity, the Banach fixed point theorem still seems to be the most important result in metric fixed point theory. Fixed point theorems are very useful in the existence theory of differential equations, integral equations, functional equations and other related areas. In 1992, Matthews [1] introduced partial metric spaces. In a partial metric space, the distance of a point from itself may not be zero. After the definition of partial metric spaces, Matthews proved the partial metric version of Banach fixed point theorem. After that, many authors were studied fixed point results in partial metric spaces. Existence of a fixed point for contraction type mappings in partially metric spaces and its applications has been considered recently by many authors [2-10]. Consistent with [1,5,11], the following definitions and results will be needed in the sequel. Throughout the article we denote $R_+ = [0, \infty)$.

Definition 1.1 [11] A partial metric on a nonempty set X is a function $p : X \times X \longrightarrow R_+$ such that for all $x, y, z \in X$:

(P_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(P_2) $p(x, x) \leq p(x, y)$,

(P_3) $p(x, y) = p(y, x)$,

(P_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

Remark 1.2 It is clear that, if $p(x, y) = 0$, then from (P_1) and (P_2) $x = y$. But if $x = y$, $p(x, y)$ may not be 0.

Example 1.3 [11] Let a function $p : R_+ \times R_+ \longrightarrow R_+$ be defined by $p(x, y) = \max\{x, y\}$ for any $x, y \in R_+$. Then (R_+, p) is a partial metric space.

Example 1.4 [11] If $X = \{[a, b] : a, b \in R, a \leq b\}$, then $p : X \times X \longrightarrow R_+$ defined by $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$ is a partial metric on X. If p is a partial metric on X, then the function $p^s : X \times X \longrightarrow R_+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \quad (1)$$

is a metric on X.

Definition 1.5 [11-13] Let (X, p) be a partial metric space. Then

(i) A sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

(ii) A sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_m, x_n)$.

(iii) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, that is $p(x, x) = \lim_{n,m\to\infty} p(x_m, x_n)$.

Remark 1.6 It is easy to see that every closed subset of a complete partial metric space is complete.
Lemma 1.7 [7,11,12] Let (X, p) be a partial metric space. Then

(a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).

(b) A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Furthermore,

$$\lim_{n\to\infty} p^s(x_n, x) = 0$$

if and only if

$$p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_m, x_n).$$

Lemma 1.8 [3] A mapping $f : X \longrightarrow X$ is said to be continuous at $a \in X$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that $f(B(a, \delta)) \subset B(f(a), \epsilon)$. The following result is easy to check.
Lemma 1.9 Let (X, p) be a partial metric space. $T : X \longrightarrow X$ is continuous if and only if given a sequence $\{x_n\} \subseteq X$ and $x \in X$ such that $p(x, x) = \lim_{n\to\infty} p(x_n, x)$, then $p(Tx, Tx) = \lim_{n\to\infty} p(Tx_n, Tx)$.
Definition 1.10 Let X be a set, T and g are selfmaps of X. A point x in X is called a coincidence point of T and g if $Tx = gx$. We shall call $w = Tx = gx$ a point of coincidence of T and g.

2. Main Result:

Theorem 2.1 Let (X, p) be a complete partial metric space and $T, g : X \longrightarrow X$ are such that $TX \subseteq gX$ and

$$G(p(Tx, Ty)) \leq \alpha G(p(gx, gy)) + \beta G(p(gy, Ty))\psi(p(gx, gy), p(gx, Tx)), \tag{2}$$

for all $x, y \in X$, where α, β are nonnegative real numbers with $\alpha + \beta < 1$, $\psi : R_+ \times R_+ \longrightarrow R_+$ is a continuous function such that $\psi(t, t) = 1$ for all $t \in R_+$ and $G : R_+ \longrightarrow R_+$ is a continuous, non- descending and subadditive function such that $G(0) = 0$. Also suppose gX is closed in (X, p). Then T and g have a unique coincidence point, that is, there exists $x \in X$ such that $Tx = gx$. Moreover, we have $p(Tx, Tx) = p(gx, gx) = 0$.
Proof. Let x_0 be an arbitrary point. Construct the sequence $\{x_n\}$ such that $Tx_n = gx_{n+1}$ for each $n = 0, 1, 2, \ldots$ which is possible since $TX \subseteq gX$. If there exists $k_0 \in N$ such that $Tx_{k_0} = Tx_{k_0+1}$, then by $Tx_n = gx_{n+1}$,

$$gx_{k_0+1} = Tx_{k_0+1},$$

that is, T and g have a coincidence at $x = x_{k_0+1}$, and so the proof is completed. So we are done in this case. Now we suppose that

$$p(Tx_n, Tx_{n+1}) > 0, \forall\, n \geq 1.$$

We claim that for all $n \in N$, we have

$$p(gx_{n+1}, gx_{n+2}) \leq \lambda^{n+1} p(gx_0, gx_1),\ where\ 0 < \lambda < 1. \tag{3}$$

By (2), we have

$$\begin{aligned}
G(p(gx_{n+1}, gx_{n+2})) &= G(p(Tx_n, Tx_{n+1})) \\
&\leq \alpha G(p(gx_n, gx_{n+1})) + \beta G(p(gx_{n+1}, Tx_{n+1}))\psi(p(gx_n, gx_{n+1}), p(gx_n, Tx_n)) \\
&= \alpha G(p(gx_n, gx_{n+1})) + \beta G(p(gx_{n+1}, gx_{n+2}))\psi(p(gx_n, gx_{n+1}), p(gx_n, gx_{n+1})) \\
&= \alpha G(p(gx_n, gx_{n+1})) + \beta G(p(gx_{n+1}, gx_{n+2}))
\end{aligned}$$

which yields that

$$G(p(gx_{n+1},gx_{n+2})) \le \frac{\alpha}{1-\beta}G(p(gx_n,gx_{n+1})).$$

Set $\lambda = \dfrac{\alpha}{1-\beta}$. Thus we have

$$\begin{aligned} &G(p(gx_{n+1},gx_{n+2})) \\ &\le \lambda G(p(gx_n,gx_{n+1})) \\ &\le \lambda^2 G(p(gx_{n-1},gx_n)) \\ &\vdots \\ &\le \lambda^{n+1}G(p(gx_0,gx_1)). \end{aligned} \tag{4}$$

Next, we claim that $\{Tx_n\}$ is a Cauchy sequence in the metric space (X,p^s). Without loss of generality assume that $n > m$. Then by using (4) and the triangle inequality for partial metrics, we have

$$\begin{aligned} &G(p(Tx_n,Tx_{n+m})) \\ &= G(p(gx_{n+1},gx_{n+m+1})) \le G(p(gx_{n+1},gx_{n+2})) + G(p(gx_{n+2},gx_{n+m+1})) \\ &\le G(p(gx_{n+1},gx_{n+2})) + G(p(gx_{n+2},gx_{n+3})) + G(p(gx_{n+3},gx_{n+m+1})). \end{aligned}$$

Inductively we have

$$\begin{aligned} 0 \le G(p(Tx_n,Tx_{n+m})) &\le G(p(gx_{n+1},gx_{n+2})) + G(p(gx_{n+2},gx_{n+3})) + ... + G(p(gx_{n+m},gx_{n+m+1})) \\ &\le (\lambda^{n+1}+\lambda^{n+2}+...+\lambda^{n+m})G(p(gx_0,gx_1)) \\ &= \lambda^{n+1}(1+\lambda+\lambda^2+...+\lambda^{m-1})G(p(gx_0,gx_1)) \\ &\le \frac{\lambda^{n+1}}{1-\lambda}G(p(gx_0,gx_1)). \end{aligned}$$

Since $\alpha+\beta<1$ then $\lambda<1$. Thus

$$\lim_{n,m\to\infty} G(p(Tx_n,Tx_m)) = G(\lim_{n,m\to\infty} p(Tx_n,Tx_m)) = 0. \tag{5}$$

We conclude that $\{Tx_n\}$ is a Cauchy sequence in (X,p). By (1), we have $p^s(Tx_n,Tx_m) \le 2p(Tx_n,Tx_m)$. Therefore

$$\lim_{n,m\to\infty} p^s(Tx_n,Tx_m) = 0. \tag{6}$$

Thus by Lemma 1.7, $\{Tx_n\}$ is a Cauchy sequence in both (X,p) and (X,p^s). Thus, there exists $z \in X$ such that $\lim_{n\to\infty} p^s(Tx_n,z) = 0$ if and only if

$$p(z,z) = \lim_{n\to\infty} p(z,Tx_n) = \lim_{n\to\infty} p(gx_{n+1},z) = \lim_{n,m\to\infty} p(Tx_m,Tx_n) = 0. \tag{7}$$

Since $\{Tx_n\} \subseteq gX$ and gX is closed, there exists $x \in X$ such that $z = gx$. Now we claim that x is a coincidence point of T and g. We have

$$\begin{aligned} G(p(gx,Tx)) &\le G(p(gx,gx_{n+1})) + G(p(gx_{n+1},Tx)) \\ &= G(p(z,gx_{n+1})) + G(p(Tx_n,Tx)), \\ &\le G(p(z,gx_{n+1})) + \alpha G(p(gx_n,gx)) + \beta G(p(gx,Tx))\psi(p(gx_n,gx),p(gx_n,Tx_n)). \end{aligned}$$

Taking $n\to\infty$ in the above inequality, we have

$$G(p(gx,Tx)) \le \beta G(p(gx,Tx))$$

which is possible only when $p(gx,Tx)=0$, which implies that $Tx=gx$, that is, x is a coincidence point of T and g. Otherwise if $p(gx,Tx)>0$, we obtain

$$G(p(gx,Tx)) \le \beta G(p(gx,Tx)) < G(p(gx,Tx))$$

a contradiction. Assume that z, y be coincidence point of T and g in X such that $y \neq z$. Then there exists t_1, t_2 in X such that $Tt_1 = gt_1 = z$ and $Tt_2 = gt_2 = y$. Using 2, we have

$$\begin{aligned} G(p(gt_1, gt_2)) = G(p(Tt_1, Tt_2)) &\leq \alpha G(p(gt_1, gt_2)) + \beta G(p(gt_2, Tt_2))\psi(p(gt_1, gt_2), p(gt_1, Tt_1)) \\ &= \alpha G(p(gt_1, gt_2)) + \beta G(p(gt_2, gt_2))\psi(p(gt_1, gt_2), p(gt_1, gt_1)). \end{aligned}$$

So

$$G(p(gt_1, gt_2)) \leq \alpha G(p(gt_1, gt_2)) < G(p(gt_1, gt_2)),$$

which is a contradiction which proves the uniqueness of point of coincidence. By taking $\psi : R_+ \times R_+ \longrightarrow R_+$ via $\psi(t, s) = \dfrac{1+s}{1+t}$ in Theorem **??**, we have the following result:

Corollary 2.2 Let (X, p) be a complete partial metric space and $T, g : X \longrightarrow X$ are such that $TX \subseteq gX$ and

$$G(p(Tx, Ty)) \leq \alpha G(p(gx, gy)) + \beta G(p(gy, Ty))\frac{1 + p(gx, Tx)}{1 + p(gx, gy)},$$

for all$x, y \in X$, where α, β are nonnegative real numbers with $\alpha + \beta < 1$ and $G : R_+ \longrightarrow R_+$ is a continuous, non-descending and subadditive function such that $G(0) = 0$. Also suppose gX is closed in (X, p). Then T and g have a unique coincidence point, that is, there exists $x \in X$ such that $Tx = gx$. Moreover, we have $p(Tx, Tx) = p(gx, gx) = 0$.

Corollary 2.3 Let (X, p) be a complete partial metric space and $T : X \longrightarrow X$ be a mapping satisfying

$$G(p(Tx, Ty)) \leq \alpha G(p(x, y)) + \beta G(p(y, Ty))\psi(p(x, y), p(x, Tx)),$$

for all$x, y \in X$, where α, β are nonnegative real numbers with $\alpha + \beta < 1$ and $\psi : R_+ \times R_+ \longrightarrow R_+$ is a continuous function such that $\psi(t, t) = 1$ for all $t \in R_+$ and $G : R_+ \longrightarrow R_+$ is a continuous, non- descending and subadditive function such that $G(0) = 0$. Then T has a unique fixed point. As a special case of Corollary 2.3, we have the following result of Matthews [1,11].

Corollary 2.4 Let (X, p) be a complete partial metric space and $T : X \longrightarrow X$ be a mapping satisfying $p(Tx, Ty) \leq \alpha p(x, y)$ for all $x, y \in X$. If $0 \leq \alpha < 1$, then T has a unique fixed point. Now, we introduced an example to support the useability of our results.

Example 2.5 Let $X = [0, 1]$ and $p(x, y) = \max\{x, y\}$, then it is clear that (X, p) is a complete partial metric space. Indeed, for any $x, y \in X$,

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) = 2\max\{x, y\} - (x + y) = |x - y|.$$

Thus, $(X, p^s) = ([0, 1], |.|)$ is the usual metric space, which is complete. Again, we define $G : R_+ \longrightarrow R_+$, $T, g : X \longrightarrow X$, by $G(t) = \dfrac{1}{3}t$, $Tx = \frac{x^3}{3x+9}$, $gx = \frac{x^2}{x+3}$ and $\psi : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ is a continuous function such that $\psi(t, t) = 1$ for all $t \in [0, \infty)$. Take $\alpha = \frac{1}{3}$ and $0 \leq \beta < \frac{2}{3}$ so that $\alpha + \beta < 1$. We show that condition (2) is satisfied.
If $x, y \in X$, then we have

$$\begin{aligned} G(p(Tx, Ty)) &= \frac{1}{3}\max\{Tx, Ty\} = \frac{1}{3}\max\left\{\frac{x^3}{3x+9}, \frac{y^3}{3y+9}\right\} \\ &\leq \frac{1}{9}\max\left\{\frac{x^2}{x+3}, \frac{y^2}{y+3}\right\} \\ &= \frac{1}{3}G(p(gx, gy)) \\ &\leq \alpha p(gx, gy) + \beta p(gy, Ty)\psi(p(gx, gy), p(gx, Tx)). \end{aligned}$$

Note that, T and g satisfy all the conditions given in Theorem 2.1. Moreover, 0 is a unique common fixed point of T and g.

Theorem 2.6 Adding to the hypotheses of Theorem 2.1 the condition, if T and g commute at their coincidence points, we obtain the uniqueness of the common fixed point of T and g.

Proof. Suppose that T and g commute at x. Set $y = Tx = gx$. Then

$$Ty = T(gx) = g(Tx) = gy, \tag{8}$$

from (2) we get

$$\begin{aligned} G(p(Tx,Ty)) &\leq \alpha G(p(gx,gy)) + \beta G(p(gy,Ty))\psi(p(gx,gy),p(gx,Tx)) \\ &= \alpha G(p(Tx,Ty)) + \beta G(p(Ty,Ty))\psi(p(Tx,Ty),p(Tx,Tx)) \\ &= \alpha G(p(Tx,Ty)) \end{aligned} \tag{9}$$

Suppose that $p(Tx,Ty) > 0$, from (9), we get

$$G(p(Tx,Ty)) \leq \alpha G(p(Tx,Ty)) < G(p(Tx,Ty)),$$

which is a contradiction. Hence $p(Tx,Ty) = 0$, that is, $p(y,Ty) = 0$. Therefore

$$Ty = gy = y. \tag{10}$$

Thus we proved that T and g have a common fixed point.
Uniqueness: Let v and w be two common fixed points of T and g. (i.e) $v = Tv = gv$ and $w = Tw = gw$. Using inequality (2.1), we have

$$\begin{aligned} G(p(w,v)) = G(p(Tw,Tv)) &\leq \alpha G(p(gw,gv)) + \beta G(p(gy,Tv))\psi(p(gw,gv),p(gw,Tw)) \\ &= \alpha G(p(w,v)), \end{aligned}$$

which is possible only when $w = v$. Hence T and g have an unique common fixed point.

References:

[1] S. G. Matthews, Partial metric topology. Research Report 212. Department of Computer Science, University of Warwick 1992.
[2] T. Abdeljawad, E. Karapınar, K. Tas, Appl. Math. Letters 24, 1900-1904, 2011.
[3] I. Altun, A. Erduran, Fixed Point Theory Appl 10 pages 2011.
[4] I. Altun, F. Sola, H. Simsek, Topology Appl. 157, 2778-2785, 2010.
[5] I. Altun, H. Simsek, J Adv Math Stud. 1, 1-8, 2008.
[6] E. Karapınar, M. E. Inci, Appl. Math. Letters 24, 1894-1899, 2008.
[7] S. Oltra, O. Valero, Rend. Istit. Mat. Univ. Trieste 36, 17-26, 2004.
[8] S. Romaguera, Fixed Point Theory Appl. 6 pages, 2010.
[9] S. Romaguera, O. Valero, Math. Structures Comput. Sci. 19, 541-563, 2009.
[10] O. Valero, Appl. Gen. Topol. 6, 229-240, 2005.
[11] S. G. Matthews, Ann. New York Acad. Sci. 728, 183-197, 1994.
[12] S. J. ONeill, Two topologies are better than one, Tech. report, University of Warwick, Coven-try, UK, http : //www.dcs.warwick.ac.uk/reports/283.html,1995.
[13] S. J. ONeill, Ann. New York Acad. Sci. 806, 304-315, 1996.