FUZZY SOFT NUMBERS

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Abstract: In this paper we introduce the notion of fuzzy soft numbers. Here defined fuzzy soft number and four arithmetic operations $\tilde{+}, \tilde{-}, \tilde{\times}, \tilde{\div}$ and related properties. Also introduce Hausdorff distance, fuzzy soft metric space, convergence sequence, Cauchy sequence, Continuity, and uniform continuity of fuzzy soft numbers. At starting of this paper, we study convex and concave fuzzy soft sets and some of their properties.

Keywords: Ideal; Fuzzy soft sets; Fuzzy soft metric space; Fuzzy soft number; Hausdorff distance.

PACS: 03E72.

1. Introduction:

In 2003, Maji et al. [1-3] studied the theory of soft set initiated by Molodtsov [4] and developed several basic notions of soft set theory. They also applied soft set theory in decision making (see [5]) and solving problems in medical, economics, engineering, etc.

In the year 1965, Zadeh [6] introduced the concept of fuzzy set theory and its applications can be found in many branches of mathematical and engineering sciences including management science, control engineering, computer science, artificial intelligence. In 1970, firstly introduced concept of fuzzy number. The theory of fuzzy number has a number of applications such as fuzzy topology, fuzzy analysis, fuzzy logic and fuzzy decision making, algebraic structures, etc. But fuzzy soft number are not developed several directions. In 2012, Das et al. 97,8] introduced important part of soft set theory, which are notion of soft real sets, soft real numbers, soft complex numbers and some of their basic properties. Also Das et al. [9] introduced a notion of soft metric and some basic properties of soft metric space.

In this paper, we study notion of fuzzy soft number. In Section 2, some preliminary results are given. In Section 3, study Convex and concave fuzzy soft sets and some of their properties. After this, we study the notion of fuzzy soft numbers and some of basic properties. In Section 4, introduce Hausdorff distance, Fuzzy soft metric space, convergence sequence, Cauchy sequence, Continuity and uniform continuity of fuzzy soft number.

2. Preliminary Results:

In this section we recall some basic concepts and definitions regarding fuzzy soft sets, fuzzy soft topology and fuzzy soft mapping.

Definition 2.1 [3] Let U be an initial universe and F be a set of parameters. Let $\tilde{P}(U)$ denote the power set of U and A be a non-empty subset of F. Then F_A is called a fuzzy soft set over U where $F : A \to \tilde{P}(U)$is a mapping from A into $\tilde{P}(U)$.

Definition 2.2 [4] F_E is called a soft set over U if and only if F is a mapping of E into the set of all subsets of the set U. In other words, the soft set is a parameterized family of subsets of the set U. Every set $F(\epsilon)$, $\epsilon \tilde{\in} E$, from this family may be considered as the set of ϵ-element of the soft set F_E or as the set of ϵ-approximate elements of the soft set.

Definition 2.3 [10] Let X be a universe and E a set of attributes. Then the collection of all fuzzy soft sets over U with attributes from E is called a fuzzy soft class and is denoted by $\overline{(X, E)}$.

Definition 2.4 [11] Let $\overline{(U, E)}$ and $\overline{(V, E')}$ be classes of hesitant fuzzy soft sets over U and V with attributes from E and E' respectively. Let $p : U \longrightarrow V$ and $q : E \longrightarrow E'$ be mappings. Then a hesitant fuzzy soft mappings $f = (p, q) : \overline{(U, E)} \longrightarrow \overline{(V, E')}$ is defined as follows;

For a hesitant fuzzy soft set F_A in $\overline{(U,E)}$, $f(F_A)$ is a hesitant fuzzy soft set in $\overline{(V,E')}$ obtained as follows: for $\beta\tilde{\in}q(E)\tilde{\subseteq}E'$ and $y\tilde{\in}V$,

$$f(F_A)(\beta)(y) = \bigcup_{\alpha\tilde{\in}q^{-1}(\beta)\cap A, s\tilde{\in}p^{-1}(y)} (\alpha)\mu_s$$

$f(F_A)$ is called a hesitant fuzzy soft image of a hesitant fuzzy soft set F_A. Hence $(F_A, f(F_A))\tilde{\in}f$, where $F_A\tilde{\subseteq}\overline{(U,E)}$, $f(F_A)\tilde{\subseteq}\overline{(V,E')}$.

Definition 2.5 [11] Let $f = (p,q) : \overline{(U,E)} \longrightarrow \overline{(V,E')}$ be a hesitant fuzzy soft mapping and G_B, a hesitant fuzzy soft set in $\overline{(V,E')}$, where $p : U \longrightarrow V, q : E \longrightarrow E'$ and $B\tilde{\subseteq}E'$. Then $f^{-1}(G_B)$ is a hesitant fuzzy soft set in $\overline{(U,E)}$ defined as follows: for $\alpha\tilde{\in}q^{-1}(B)\tilde{\subseteq}E$ and $x\tilde{\in}U$,

$$f^{-1}(G_B)(\alpha)(x) = (q(\alpha))\mu_{p(x)}$$

$f^{-1}(G_B)$ is called a hesitant fuzzy soft inverse image of G_B.

Definition 2.6 [12] A fuzzy soft topology τ on (U,E) is a family of fuzzy soft sets over (U,E) satisfying the following properties

(i) $\tilde{\phi}, \tilde{E}\tilde{\in}\tau$

(ii) if $F_A, G_B\tilde{\in}\tau$, then $F_A\tilde{\cap}G_B\tilde{\in}\tau$.

(iii) if $F_{A_\alpha}\tilde{\in}\tau$ for all $\alpha\tilde{\in}\Delta$ an index set, then $\bigcup_{\alpha\in\Delta} F_{A_\alpha}\tilde{\in}\tau$.

Definition 2.7 [12] If τ is a fuzzy soft topology on (U,E), the triple (U,E,τ) is said to be a fuzzy soft topological space. Also each member of τ is called a fuzzy soft open set in (U,E,τ).

Definition 2.8 [13] Let (U,E,τ) be a fuzzy soft topological space. Let F_A be a fuzzy soft set over (U,E). The fuzzy soft closure of F_A is defined as the intersection of all fuzzy soft closed sets which contained F_A and is denoted by $\bar{F_A}$ or $cl(F_A)$ we write

$$cl(F_A) = \tilde{\bigcap}\{G_B : G_B \text{ is fuzzy soft closed and } F_A\tilde{\subseteq}G_B\}.$$

Definition 2.9 [14] Let $\overline{(U,E)}$ and $\overline{(V,E')}$ be classes of fuzzy soft sets over U and V with attributes from E and E' respectively. Let $p : U \longrightarrow V$ and $q : E \longrightarrow E'$ be two mappings. Then $f = (p,q) : \overline{(U,E)} \longrightarrow \overline{(V,E')}$ is called a fuzzy soft mappings from $\overline{(U,E)}$ to $\overline{(V,E')}$.
If p and q is injective then the fuzzy soft mapping $f = (p,q)$ is said to be injective.
If p and q is surjective then the fuzzy soft mapping $f = (p,q)$ is said to be surjective.
If p and q is constant then the fuzzy soft mapping $f = (p,q)$ is said to be constant.

Definition 2.10 [15] Let (U,E,τ_1) and (U,E,τ_2) be two fuzzy soft topological spaces.

(i) A fuzzy soft mapping $f = (p,q) : \overline{(U,E,\tau_1)} \longrightarrow \overline{(U,E,\tau_2)}$ is called fuzzy soft continuous if $f^{-1}(G_B)\tilde{\in}\tau_1, \forall G_B\tilde{\in}\tau_2$.

(ii) A fuzzy soft mapping $f = (p,q) : \overline{(U,E,\tau_1)} \longrightarrow \overline{(U,E,\tau_2)}$ is called fuzzy soft open if $f(F_A)\tilde{\in}\tau_2, \forall F_A\tilde{\in}\tau_1$.

Definition 2.11 [16] Let (U,E,τ_1) and (U,E,τ_2) be two soft topological space. Then a soft multifunction $f : \overline{(U,E,\tau_1)} \longrightarrow \overline{(U,E,\tau_2)}$ is said to be

(i) soft upper semi continuous at a soft point $e_i(F_A)\tilde{\in}(U,E)$if for every soft open set $G_B\tilde{\in}(V,E)$ such that $f(e_i(F_A))\tilde{\subseteq}G_B$, there exists a soft semi open neighborhood H_A of $e_i(F_A)$ such that $f(e_i(H_A))\tilde{\subseteq}G_B$, $\forall f(e_i(H_A))\tilde{\in}H_A$

(ii) soft lower semi continuous at a soft point $e_i(F_A)\tilde{\in}(U,E)$if for every soft open set $G_B\tilde{\in}(V,E)$ such that $f(e_i(F_A))\tilde{\cap}G_B\tilde{\neq}\tilde{\phi}$, there exists a soft semi open neighborhood H_A of $e_i(F_A)$ such that $f(e_i(H_A))\tilde{\cap}G_B\tilde{\neq}\tilde{\phi}$, $\forall f(e_i(H_A))\tilde{\in}H_A$

(iii) soft upper(lower) semi continuous if f has this property at every soft point of (U,E).

3. Fuzzy soft numbers:

Definition 3.1 Let $F_A = \{F(e_i) = (h_t, \mu_{F(e_i)}(h_t)); h_t\tilde{\in}U; t = 1,2,...m; i = 1,2,...,n\}$ be a fuzzy soft set in (U,E). Now convert object sets h_t are integers namely $h_t = i, i = 1,2,...,n$. Then F_A is called convex fuzzy soft set if and only if membership function $\mu_{F(e_i)}$ satisfies following conditions:

(i) each parameter e_i of F_A, $\mu_{F(e_i)}(\lambda.h_1 + (1-\lambda).h_2) \geq \min\{\mu_{F(e_i)}(h_1), \mu_{F(e_i)}(h_2)\}$ where $\lambda\tilde{\in}[0,1]$. and $h_1, h_2\tilde{\in}R$.

(ii) $\mu_{\cap_i F(e_i)}(\lambda.h_1 + (1-\lambda).h_2) \geq \min\{\mu_{\cap_i F(e_i)}(h_1), \mu_{\cap_i F(e_i)}(h_2)\}$ where $\lambda\tilde{\in}[0,1]$. and $h_1, h_2\tilde{\in}R$.

Otherwise it is non-convex fuzzy soft set.

Example 3.2 Let

$$P_A = \{P(e_1) = \{(h_1, 0.1), (h_2, 0.6), (h_3, 1.0), (h_4, 0.8), (h_5, 0.2)\}$$
$$P(e_2) = \{(h_1, 0.3), (h_2, 0.9), (h_3, 1.0), (h_4, 0.7), (h_5, 0.2)\}\}.$$

Therefore P_A be a convex fuzzy soft set.

Theorem 3.3 If F_A and G_A are convex fuzzy soft sets then $F_A\tilde{\cap}G_A$ is convex fuzzy soft set.

Proof. Let F_A and G_A are convex fuzzy soft sets in (U, E). Let $h_1, h_2\tilde{\in}U$ and $e_1, e_2\tilde{\in}E$. Now convert objects h_1, h_2 are integers namely 1,2. Therefore

$$\mu_{F(e_1\tilde{\cap}e_2)}(\lambda.h_1 + (1-\lambda).h_2) \geq \min\{\mu_{F(e_1\tilde{\cap}e_2)}(h_1), \mu_{F(e_1\tilde{\cap}e_2)}(h_2)\}$$
and
$$\mu_{G(e_1\tilde{\cap}e_2)}(\lambda.h_1 + (1-\lambda).h_2) \geq \min\{\mu_{G(e_1\tilde{\cap}e_2)}(h_1), \mu_{G(e_1\tilde{\cap}e_2)}(h_2)\},$$

where $\lambda\tilde{\in}[0,1]$ and $h_1, h_2\tilde{\in}R$.
Now

$$\begin{aligned}
&\mu_{F(e_1\tilde{\cap}e_2)\tilde{\cap}G(e_1\tilde{\cap}e_2)}(\lambda.h_1 + (1-\lambda)h_2)\\
&= \mu_{F(e_1\tilde{\cap}e_2)}(\lambda.h_1 + (1-\lambda).h_2)\tilde{\cap}\mu_{G(e_1\tilde{\cap}e_2)}(\lambda.h_1 + (1-\lambda).h_2)\\
&\geq \min\{\mu_{F(e_1\tilde{\cap}e_2)}(h_1), \mu_{F(e_1\tilde{\cap}e_2)}(h_2)\}\tilde{\cap}\min\{\mu_{G(e_1\tilde{\cap}e_2)}(h_1), \mu_{G(e_1\tilde{\cap}e_2)}(h_2)\}\\
&\geq \min\{\mu_{F(e_1\tilde{\cap}e_2)\tilde{\cap}G(e_1\tilde{\cap}e_2)}(h_1), \mu_{F(e_1\tilde{\cap}e_2)\tilde{\cap}G(e_1\tilde{\cap}e_2)}(h_2)\} \text{ where } \lambda\tilde{\in}[0,1] \text{ and } h_1, h_2\tilde{\in}R.
\end{aligned}$$

Hence proved.

Theorem 3.4 If F_A and G_A are convex fuzzy soft sets and $F_A\tilde{\subseteq}G_A$ then $F_A\tilde{\cup}G_A$ and $F_A\tilde{\cap}G_A$ are convex fuzzy soft set.

Proof. Proof of the results are obvious.

Theorem 3.5 The union of any family of convex fuzzy soft sets is not necessarily a convex fuzzy soft set.

Proof. The proof is straightforward.

Definition 3.6 Let $F_A = \{F(e_i) = (h_t, \mu_{F(e_i)}(h_t)); h_t\tilde{\in}U; t = 1, 2, ...m; i = 1, 2, ..., n\}$ be a fuzzy soft set in (U, E). Then F_A is called concave fuzzy soft set if and only if membership function $\mu_{F(e_i)}$ satisfies following conditions:

(i) each parameter e_i of F_A,
$\mu_{F(e_i)}(\lambda.h_1 + (1-\lambda).h_2) \leq \max\{\mu_{F(e_i)}(h_1), \mu_{F(e_i)}(h_2)\}$ where $\lambda\tilde{\in}[0,1]$ and $h_1, h_2\tilde{\in}R$.

(ii) $\mu_{\cap_i F(e_i)}(\lambda.h_1 + (1-\lambda).h_2) \leq \max\{\mu_{\cap_i F(e_i)}(h_1), \mu_{\cap_i F(e_i)}(h_2)\}$ where $\lambda\tilde{\in}[0,1]$. and $h_1, h_2\tilde{\in}R$.

Otherwise it is non-concave fuzzy soft set.

Example 3.7 Let

$$N_A = \{N(e_1) = \{(h_1, 0.9), (h_2, 0.4), (h_3, 0.0), (h_4, 0.2), (h_5, 0.8)\}$$
$$N(e_2) = \{(h_1, 0.7), (h_2, 0.1), (h_3, 0.0), (h_4, 0.3), (h_5, 0.8)\}\}.$$

Therefore N_A be a concave fuzzy soft set.

Proposition 3.8 If F_A and G_A are concave fuzzy soft sets. $F_A\tilde{\cap}G_A$ and $F_A\tilde{\cup}G_A$ are concave fuzzy soft sets.

Proof. Let F_A and G_A are concave fuzzy soft sets in (U, E). Let $h_1, h_2\tilde{\in}U$ and $e_1, e_2\tilde{\in}E$. Now convert objects h_1, h_2 are integers namely 1,2. Therefore

$$\mu_{F(e_1\tilde{\cap}e_2)}(\lambda.h_1 + (1-\lambda).h_2) \leq \max\{\mu_{F(e_1\tilde{\cap}e_2)}(h_1), \mu_{F(e_1\tilde{\cap}e_2)}(h_2)\}$$
and
$$\mu_{G(e_1\tilde{\cap}e_2)}(\lambda.h_1 + (1-\lambda).h_2) \leq \max\{\mu_{G(e_1\tilde{\cap}e_2)}(h_1), \mu_{G(e_1\tilde{\cap}e_2)}(h_2)\},$$

where $\lambda\tilde{\in}[0,1]$ and $h_1, h_2\tilde{\in}R$.
Now

$$\begin{aligned}&\mu_{F(e_1\tilde{\cap}e_2)\tilde{\cap}G(e_1\tilde{\cap}e_2)}(\lambda.h_1+(1-\lambda)h_2)\\&=\mu_{F(e_1\tilde{\cap}e_2)}(\lambda.h_1+(1-\lambda).h_2)\tilde{\cap}\mu_{G(e_1\tilde{\cap}e_2)}(\lambda.h_1+(1-\lambda).h_2)\\&\leq\max\{\mu_{F(e_1\tilde{\cap}e_2)}(h_1),\mu_{F(e_1\tilde{\cap}e_2)}(h_2)\}\tilde{\cap}\max\{\mu_{G(e_1\tilde{\cap}e_2)}(h_1),\mu_{G(e_1\tilde{\cap}e_2)}(h_2)\}\\&\leq\max\{\mu_{F(e_1\tilde{\cap}e_2)\tilde{\cap}G(e_1\tilde{\cap}e_2)}(h_1),\mu_{F(e_1\tilde{\cap}e_2)\tilde{\cap}G(e_1\tilde{\cap}e_2)}(h_2)\}\text{ where }\lambda\tilde{\in}[0,1]\text{ and }h_1,h_2\tilde{\in}R.\end{aligned}$$

Hence $F_A\tilde{\cap}G_A$ be a concave fuzzy soft sets.
Similarly we prove that $F_A\tilde{\cup}G_A$ be a concave fuzzy soft set.

Proposition 3.9 Let F_A be a convex fuzzy soft set then F_A^C be a concave fuzzy soft sets.

Proof. Let F_A be a convex fuzzy soft set. Then

(i) each parameter e_i of F_A, $\mu_{F(e_i)}(\lambda.h_1+(1-\lambda).h_2)\geq\min\{\mu_{F(e_i)}(h_1),\mu_{F(e_i)}(h_2)\}$ where $\lambda\tilde{\in}[0,1]$. and $h_1,h_2\tilde{\in}R$.

(ii) $\mu_{\cap_iF(e_i)}(\lambda.h_1+(1-\lambda).h_2)\geq\min\{\mu_{\cap_iF(e_i)}(h_1),\mu_{\cap_iF(e_i)}(h_2)\}$ where $\lambda\tilde{\in}[0,1]$ and $h_1,h_2\tilde{\in}R$.

Now, (i) Each parameter e_i of F_A^C,

$$\begin{aligned}&\mu_{F^C(e_i)}(\lambda.h_1+(1-\lambda).h_2)\\&=1-\mu_{F(e_i)}(\lambda.h_1+(1-\lambda).h_2)\\&\leq1-\min\{\mu_{F(e_i)}(h_1),\mu_{F(e_i)}(h_2)\}\\&\leq\max\{1-\mu_{F(e_i)}(h_1),1-\mu_{F(e_i)}(h_2)\}\\&\leq\max\{\mu_{F^C(e_i)}(h_1),\mu_{F^C(e_i)}(h_2)\},\end{aligned}$$

where $\lambda\tilde{\in}[0,1]$. and $h_1,h_2\tilde{\in}R$.
(ii) Similarly we prove that

$$\mu_{\cap_iF^C(e_i)}(\lambda.h_1+(1-\lambda).h_2)\leq\max\{\mu_{\cap_iF^C(e_i)}(h_1),\mu_{\cap_iF^C(e_i)}(h_2)\}$$

Hence F_A^C be a concave fuzzy soft sets.

Proposition 3.10 If F_A and G_A are convex and concave fuzzy soft set, respectively and $F_A\tilde{\subseteq}G_A$. Then $F_A\tilde{\cup}G_A$ and $F_A\tilde{\cap}G_A$ are concave and convex fuzzy soft set, respectively.

Proof. Since $F_A\tilde{\subseteq}G_A$, therefore we have

$$F_A\tilde{\cup}G_A=G_A,$$

which is concave fuzzy soft set. Then $F_A\tilde{\cap}G_A$ is a convex fuzzy soft set.

Definition 3.11 A fuzzy soft set $F_A=\{F(e_i)=(h_t,\mu_{F(e_i)}(h_t));h_t\tilde{\in}U;t=1,2,...m;i=1,2,...,n\}$ is called a normalized fuzzy soft set if it satisfied following two conditions:

(i) there is at least one point $h_t\tilde{\in}U$ with $\mu_{F(e_i)}(h_t)=1$ for each e_i.

(ii) there is at least one point $h_t\tilde{\in}U$ with $\mu_{F(\cap_i(e_i))}(h_t)=1$ for $\cap_i(e_i)$.

Otherwise it is non-normalized.

Example 3.12 Let

$$\begin{aligned}K_A=\{K(e_1)&=\{(h_1,0.2),(h_2,1.0),(h_3,0.3)\},\\K(e_2)&=\{(h_1,0.1),(h_2,1.0),(h_3,0.2)\}\}.\end{aligned}$$

Therefore K_A is a normalized fuzzy soft set.

Definition 3.13 A fuzzy soft set $F_A=\{F(e_i)=(h_t,\mu_{F(e_i)}(h_t));h_t\tilde{\in}U;t=1,2,...m;i=1,2,...,n\}$ is a fuzzy soft number if its membership functions $\mu_{F(e_i)}$ is

(i) fuzzy soft convex;

(ii) fuzzy soft normalized;

(iii) fuzzy soft upper semi-continuous.

(iv) $cl\{h_t; \mu_{F(e_i)}(h_t) > 0\}$ is fuzzy soft compact.

Proposition 3.14 Fuzzy soft numbers always normalized at same object of each parameters.

Proof. Suppose F_A be a fuzzy soft number normalized at two objects h_2 and h_3 of parameters e_1 and e_2 respectively. Therefore

$$\mu_{F(e_1\tilde{\cap}e_2)}(h_2) \neq 1 \text{ and } \mu_{F(e_1\tilde{\cap}e_2)}(h_3) \neq 1.$$

Therefore F_A is not a fuzzy soft number.
Hence Fuzzy soft numbers always normalized at same object of each parameters.

Proposition 3.15 Complement of fuzzy soft numbers is a concave fuzzy soft sets.

Proof. Follows from the Proposition 3.9.

Definition 3.16 Let $F_A = \{F(e_i) = (h_t, \mu_{F(e_i)}(h_t)); h_t\tilde{\in}U; t = 1,2,...m; i = 1,2,...,n\}$ and $G_A = \{G(e_i) = (h_t, \mu_{G(e_i)}(h_t)); h_t\tilde{\in}U; t = 1,2,...m; i = 1,2,...,n\}$ are two fuzzy soft numbers. Here we denote μ_F and μ_G are grade membership of objects of fuzzy soft sets F_A and G_A respectively. Then four arithmetic operations for fuzzy soft numbers are defined as follows:
For $t = 1,2,...,m$

(i) $F_A\tilde{+}G_A = \{F(e_i)\tilde{+}G(e_i)\} = \{(h_t, \mu_F + \mu_G - \mu_F.\mu_G)\}$.

(ii) $F_A\tilde{-}G_A = \{F(e_i)\tilde{-}G(e_i)\} = \{(h_t, \mu_F.\mu_G)\}$.

(iii) $F_A\tilde{\times}G_A = \{F(e_i)\tilde{\times}G(e_i)\} = \left\{\left(h_t, \frac{\mu_F.\mu_G}{\max(\mu_F,\mu_G)}\right)\right\}$.

(iv) $F_A\tilde{\div}G_A = \{F(e_i)\tilde{\div}G(e_i)\} = \left\{\left(h_t, \frac{\mu_F}{\max(\mu_F,\mu_G)}\right)\right\}$.

Example 3.17 Let

$$\begin{aligned} F_A = \{F(e_1) &= \{(h_1, 0.0), (h_2, 0.6), (h_3, 1.0), (h_4, 0.8), (h_5, 0.1)\},\\ F(e_2) &= \{(h_1, 0.1), (h_2, 0.7), (h_3, 1.0), (h_4, 0.8), (h_5, 0.0)\}\} \end{aligned}$$

and

$$\begin{aligned} G_A = \{G(e_1) &= \{(h_1, 0.1), (h_2, 0.8), (h_3, 1.0), (h_4, 0.6), (h_5, 0.0)\},\\ G(e_2) &= \{(h_1, 0.3), (h_2, 0.9), (h_3, 1.0), (h_4, 0.8), (h_5, 0.2)\}\} \end{aligned}$$

are two fuzzy soft numbers. Then

(i) $F_A\tilde{+}G_A = \{e_1 = \{(h_1, 0.10), (h_2, 0.92), (h_3, 1.00), (h_4, 0.92), (h_5, 0.10)\},$
$e_2 = \{(h_1, 0.37), (h_2, 0.97), (h_3, 1.00), (h_4, 0.96), (h_5, 0.10)\}\}$

(ii) $F_A\tilde{-}G_A = \{e_1 = \{(h_1, 0.00), (h_2, 0.48), (h_3, 1.00), (h_4, 0.48), (h_5, 0.00)\},$
$e_2 = \{(h_1, 0.03), (h_2, 0.63), (h_3, 1.00), (h_4, 0.64), (h_5, 0.00)\}\}$

(iii) $F_A\tilde{\times}G_A = \{e_1 = \{(h_1, 0.00), (h_2, 0.60), (h_3, 1.00), (h_4, 0.60), (h_5, 0.00)\},$
$e_2 = \{(h_1, 0.10), (h_2, 0.70), (h_3, 1.00), (h_4, 0.80), (h_5, 0.00)\}\}$

(iv) $F_A\tilde{\div}G_A = \{e_1 = \{(h_1, 0.00), (h_2, 0.75), (h_3, 1.00), (h_4, 1.00), (h_5, 1.00)\},$
$e_2 = \{(h_1, 0.34), (h_2, 0.78), (h_3, 1.00), (h_4, 1.00), (h_5, 1.00)\}\}$.

Proposition 3.18 If membership value of same objects of both soft numbers are zero, then multiplication and division operations of that soft numbers are undefined.

Proof. Follows from definition.

Proposition 3.19 Let F_A and G_A are two fuzzy soft numbers, then $F_A\tilde{+}G_A$, $F_A\tilde{-}G_A$ and $F_A\tilde{\times}G_A$ are also fuzzy soft numbers.

Proof. Let $F_A = \{F(e_i) = (h_t, \mu_{F(e_i)}(h_t))\}$ and $G_A = \{G(e_i) = (h_t, \mu_{G(e_i)}(h_t))\}$ are two fuzzy soft numbers. Therefore

$$F_A\tilde{+}G_A = \{F(e_i)\tilde{+}G(e_i)\} = \{(h_t, \mu_F + \mu_G - \mu_F.\mu_G)\}$$

and

$$F_A\tilde{-}G_A = \{F(e_i)\tilde{-}G(e_i)\} = \{(h_t, \mu_F.\mu_G)\}.$$

Since membership functions μ_F and μ_G are convex and normalized. Therefore $\mu_F + \mu_G - \mu_F.\mu_G$ and $\mu_F.\mu_G$ are convex and normalized. Also $\mu_F + \mu_G - \mu_F.\mu_G$ and $\mu_F.\mu_G$ are upper semi-continuous and closure of that values(> 0) are fuzzy soft compact. Hence $F_A\tilde{+}G_A$, $F_A\tilde{-}G_A$ are fuzzy soft numbers.
Now Consider $F_A\tilde{\times}G_A$. Since

$$F_A\tilde{\times}G_A = \{F(e_i)\tilde{\times}G(e_i)\} = \left\{\left(h_t, \frac{\mu_F.\mu_G}{\max(\mu_F, \mu_G)}\right)\right\}.$$

Case 1: If $F_A\tilde{\subseteq}G_A$,then $F_A\tilde{\times}G_A = \{(h_t, \mu_F)\} = F_A$.
Case 2: If $G_A\tilde{\subseteq}F_A$,then $F_A\tilde{\times}G_A = \{(h_t, \mu_G)\} = G_A$.
Case 3: If $F_A = G_A$,then $F_A\tilde{\times}G_A = G_A = F_A$.
Hence $F_A\tilde{\times}G_A$ is fuzzy soft number.

Proposition 3.20 Let F_A and G_A are two fuzzy soft numbers, then

(i) $F_A\tilde{+}G_A = G_A\tilde{+}F_A$.

(ii) $F_A\tilde{\times}G_A = G_A\tilde{\times}F_A$.

(iii) $F_A\tilde{+}\tilde{\phi} = F_A$.

(iv) $F_A\tilde{\times}\tilde{E} = F_A$.

(v) $F_A\tilde{\times}\tilde{\phi} = \tilde{\phi}$.

(vi) $\tilde{\phi}\tilde{\div}F_A = \tilde{\phi}$.

(vii) $F_A\tilde{\div}\tilde{\phi} = \tilde{E}$.

(viii) $F_A\tilde{\div}F_A = \tilde{E}$.

Proof. Proof of the results are obvious.

Proposition 3.21 Let F_A, G_A and H_A are three fuzzy soft numbers, then

(i) $(F_A\tilde{+}G_A)\tilde{+}H_A = F_A\tilde{+}(G_A\tilde{+}H_A)$.

(ii) $(F_A\tilde{\times}G_A)\tilde{\times}H_A = F_A\tilde{\times}(G_A\tilde{\times}H_A)$.

(iii) $F_A\tilde{+}(G_A\tilde{\cup}H_A) = (F_A\tilde{+}G_A)\tilde{\cup}(F_A\tilde{+}H_A)$.

(iv) $F_A\tilde{+}(G_A\tilde{\cap}H_A) = (F_A\tilde{+}G_A)\tilde{\cap}(F_A\tilde{+}H_A)$.

(v) $F_A\tilde{-}(G_A\tilde{\cup}H_A) = (F_A\tilde{-}G_A)\tilde{\cap}(F_A\tilde{-}H_A)$.

(vi) $F_A\tilde{-}(G_A\tilde{\cap}H_A) = (F_A\tilde{-}G_A)\tilde{\cup}(F_A\tilde{-}H_A)$.

(vii) $(G_A\tilde{\cup}H_A)\tilde{-}F_A = (G_A\tilde{-}F_A)\tilde{\cup}(H_A\tilde{-}F_A)$.

(viii) $(G_A\tilde{\cap}H_A)\tilde{-}F_A = (G_A\tilde{-}F_A)\tilde{\cap}(H_A\tilde{-}F_A)$.

(ix) $F_A\tilde{\times}(G_A\tilde{\cup}H_A) = (F_A\tilde{\times}G_A)\tilde{\cup}(F_A\tilde{\times}H_A)$.

(x) $F_A\tilde{\times}(G_A\tilde{\cap}H_A) = (F_A\tilde{\times}G_A)\tilde{\cap}(F_A\tilde{\times}H_A)$.

(xi) $F_A\tilde{\div}(G_A\tilde{\cup}H_A) = (F_A\tilde{\div}G_A)\tilde{\cap}(F_A\tilde{\div}H_A)$.

(xii) $F_A\tilde{\div}(G_A\tilde{\cap}H_A) = (F_A\tilde{\div}G_A)\tilde{\cup}(F_A\tilde{\div}H_A)$.

(xiii) $(G_A\tilde{\cup}H_A)\tilde{\div}F_A = (G_A\tilde{\div}F_A)\tilde{\cup}(H_A\tilde{\div}F_A)$.

(xiv) $(G_A\tilde{\cap}H_A)\tilde{\div}F_A = (G_A\tilde{\div}F_A)\tilde{\cap}(H_A\tilde{\div}F_A)$.

Proof. Proof of the results are obvious.

4. Hausdorff distance between two fuzzy soft numbers:

Definition 4.1 Let F_A and G_A are two fuzzy soft numbers. The Hausdorff distance between F_A and G_A is defined as

$$\tilde{d}(F_A, G_A) = \max\{\mid \mu_{F(\cap_i(e_i))}(h_t) - \mu_{G(\cap_i(e_i))}(h_t) \mid\},$$

where $i = 1, 2, ..., n; t = 1, 2, ..., m$.

Example 4.2 From Example 3.17

$$\begin{aligned}\tilde{d}(F_A, G_A) &= \max\{\mid 0.0 - 0.1 \mid, \mid 0.6 - 0.8 \mid, \mid 1.0 - 1.0 \mid, \mid 0.8 - 0.6 \mid, \mid 0.0 - 0.0 \mid\} \\ &= \max\{0.1, 0.2, 0.0, 0.2, 0.0\} = 0.2\end{aligned}$$

Definition 4.3 Consider a fuzzy soft point $e_i(F_A); (i = 1, 2, ..n)$ in a soft number F_A and G_A be any fuzzy soft number. The Hausdorff distance between soft point and G_A is defined by

$$\tilde{d}(e_i(F_A), G_A) = \max\{\mid \mu_{e_i(F_A)}(h_t) - \mu_{G(\cap_i(e_i))}(h_t) \mid\},$$

where $i = 1, 2, ..., n; t = 1, 2, ..., m$.

Example 4.4 From Example 3.17 and consider a soft point $e_1(F_A)$ in a soft number F_A as

$$e_1(F_A) = \{(h_1, 0.0), (h_2, 0.6), (h_3, 1.0), (h_4, 0.8), (h_5, 0.1)\}.$$

Therefore

$$\begin{aligned}\tilde{d}(e_1(F_A), F_A) &= \max\{\mid 0.0 - 0.0 \mid, \mid 0.6 - 0.6 \mid, \mid 1.0 - 1.0 \mid, \mid 0.8 - 0.8 \mid, \mid 0.1 - 0.0 \mid\} \\ &= \max\{0.0, 0.0, 0.0, 0.0, 0.1\} = 0.1\end{aligned}$$

and

$$\begin{aligned}\tilde{d}(e_1(F_A), G_A) &= \max\{\mid 0.0 - 0.1 \mid, \mid 0.6 - 0.8 \mid, \mid 1.0 - 1.0 \mid, \mid 0.8 - 0.6 \mid, \mid 0.1 - 0.0 \mid\} \\ &= \max\{0.1, 0.2, 0.0, 0.2, 0.1\} = 0.2.\end{aligned}$$

Proposition 4.5 Distance between soft number and complement of soft number always one.

Proof. Consider the fuzzy soft number F_A. Therefore

$$\begin{aligned}\tilde{d}(F_A, F_A^C) &= \max\{\mid 0.0 - 0.9 \mid, \mid 0.6 - 0.3 \mid, \mid 1.0 - 0.0 \mid, \mid 0.8 - 0.2 \mid, \mid 0.0 - 0.9 \mid\} \\ &= \max\{0.9, 0.3, 1.0, 0.6, 0.9\} = 1.0.\end{aligned}$$

Theorem 4.6 If L_A, M_A, N_A are three fuzzy soft numbers and $L_A \tilde{\subseteq} M_A \tilde{\subseteq} H_A$. Then

(i) $\tilde{d}(L_A, M_A) \tilde{\leq} \tilde{d}(L_A, H_A)$.

(ii) $\tilde{d}(M_A, H_A) \tilde{\leq} \tilde{d}(L_A, H_A)$.

Proof. Consider

$$\begin{aligned}L_A = \{L(e_1) &= \{(h_1, 0.2), (h_2, 0.8), (h_3, 1.0), (h_4, 0.7), (h_5, 0.1)\}, \\ L(e_2) &= \{(h_1, 0.3), (h_2, 0.9), (h_3, 1.0), (h_4, 0.7), (h_5, 0.2)\}\};\end{aligned}$$

$$\begin{aligned}M_A = \{M(e_1) &= \{(h_1, 0.1), (h_2, 0.8), (h_3, 1.0), (h_4, 0.6), (h_5, 0.0)\} \\ M(e_2) &= \{(h_1, 0.1), (h_2, 0.7), (h_3, 1.0), (h_4, 0.7), (h_5, 0.0)\}\}\end{aligned}$$

and

$$\begin{aligned}H_A = \{H(e_1) &= \{(h_1, 0.1), (h_2, 0.7), (h_3, 1.0), (h_4, 0.6), (h_5, 0.0)\} \\ H(e_2) &= \{(h_1, 0.0), (h_2, 0.7), (h_3, 1.0), (h_4, 0.7), (h_5, 0.0)\}\}.\end{aligned}$$

Therefore

$$\tilde{d}(L_A, M_A) = 0.1; \tilde{d}(L_A, H_A) = 0.2; \tilde{d}(M_A, H_A) = 0.1.$$

Hence $\tilde{d}(L_A, M_A) \tilde{\leq} \tilde{d}(L_A, H_A)$ and $\tilde{d}(M_A, H_A) \tilde{\leq} \tilde{d}(L_A, H_A)$.

Definition 4.7 Let F_A be a fuzzy soft number over (U, E). A mapping $\tilde{d} : (U, E) \times (U, E) \longrightarrow [0, 1]$ is said to be a fuzzy soft metric of soft numbers over (U, E), if $\tilde{d}$ satisfies the following conditions:

(i) $\tilde{d}(F_A, G_A)\tilde{\geq}0, \forall F_A, G_A\tilde{\in}(U, E)$,

(ii) $\tilde{d}(F_A, G_A) = 0 \Longleftrightarrow F_A = G_A$,

(iii) $\tilde{d}(F_A, G_A) = \tilde{d}(G_A, F_A)$,

(iv) $\tilde{d}(F_A, G_A)\tilde{\leq}\tilde{d}(F_A, G_A) + \tilde{d}(G_A, H_A)$.

Then the triple $(U, E, \tilde{d})$ is called a fuzzy soft metric space.

Definition 4.8 Let $(U, E, \tilde{d})$ be a fuzzy soft metric space. Then the diameter of soft number F_A is denoted by

$$\tilde{\delta}(F_A) = \max\{\tilde{d}(e_i(F_A), e_j(F_A))\};\ \text{for each}\ e_i, e_j\tilde{\in}F_A.$$

Proposition 4.9 Diameter of each fuzzy soft point number is always zero.

Proof. We know that distance between each fuzzy soft point is zero. Therefore maximum distance between each soft point is also zero. Hence proved the result.

Theorem 4.10 If $F_A\tilde{\subseteq}G_A$ of (U, E), then $\tilde{\delta}(F_A)\tilde{\leq}\tilde{\delta}(G_A)$.

Proof. Since $F_A\tilde{\subseteq}G_A$, therefore we have

$$e_i(F_A)\tilde{\subseteq}e_i(G_A), e_j(F_A)\tilde{\subseteq}e_j(G_A), \forall e_i, e_j\tilde{\in}F_A, G_A.$$

Now

$$\tilde{\delta}(F_A) = \max\{\tilde{d}(e_i(F_A), e_j(F_A))\}\tilde{\leq}\max\{\tilde{d}(e_i(G_A), e_j(G_A))\}\tilde{\leq}\tilde{d}(G_A)$$

for each $e_i, e_j\tilde{\in}F_A, G_A$. This completes the proof.

Theorem 4.11 Let $F_A, G_A\tilde{\in}(U, E)$ and $F_A\tilde{\cap}G_A\tilde{\neq}\tilde{\phi}$. Then $\tilde{\delta}(F_A\tilde{\cup}G_A)\tilde{\leq}\tilde{\delta}(F_A)\tilde{+}\tilde{\delta}(G_A)$.

Proof. Follows from definition.

Definition 4.12 Let $(U, E, \tilde{d})$ be a fuzzy soft metric space and $F_A\tilde{\in}(U, E)$. Then for any $r\tilde{\in}(0, 1)$, the set

$$S_r(F_A) = \{G_A\tilde{\in}(U, E) : \tilde{d}(F_A, G_A)\tilde{<}r\}$$

is called a fuzzy soft open sphere of radius $'r'$ centered at F_A.

Definition 4.13 Let $(U, E, \tilde{d})$ be a fuzzy soft metric space and $F_A\tilde{\in}(U, E)$. Then for any $r\tilde{\in}(0, 1)$, the set

$$S_r[F_A] = \{G_A\tilde{\in}(U, E) : \tilde{d}(F_A, G_A)\tilde{\leq}r\}$$

is called fuzzy soft closed sphere of radius $'r'$ centered at F_A.

Definition 4.14 Let $(U, E, \tilde{d})$ be a fuzzy soft metric space and $F_A\tilde{\in}(U, E)$. A sub class (V', F) of (U, E) is called a fuzzy soft neighborhood of a soft number $F_A\tilde{\in}(U, E)$, we denoted as $N_{F_A}^{(V',F)}$, if there exists an open sphere of fuzzy soft number $S_r(F_A)$ center at F_A and contained in (V', F). i.e. $S_r(F_A)\tilde{\subseteq}N_{F_A}^{(V',F)}$, for some $r\tilde{\in}(0, 1)$.

Definition 4.15 A sub class (V', F) of a fuzzy soft metric space $(U, E, \tilde{d})$ is said to be open in $(U, E, \tilde{d})$, if (V', F) is a fuzzy soft neighborhood of each of its soft numbers. i.e.if for each $F_A\tilde{\in}(V', F)$, there is an $r\tilde{\in}(0, 1)$ such that $S_r(F_A)\tilde{\subseteq}(V', F)$.

Definition 4.16 Let $\{(F_A)_n\}$ be a sequence of fuzzy soft numbers in a fuzzy soft metric space $(U, E, \tilde{d})$. The sequence $\{(F_A)_n\}$ is said to converge in $(U, E, \tilde{d})$ if there a fuzzy soft number F_A' in (U, E) such that $\tilde{d}((F_A)_n, F_A')\tilde{\to}0$ as $n\tilde{\to}\infty$.

That is for every $\varepsilon\tilde{>}0$ there exists a positive integer m such that

$$\tilde{d}((F_A)_n, F_A')\tilde{\leq}\varepsilon,\ \forall n \geq m.$$

It is denoted as $\lim_{n\to\infty}(F_A)_n = F_A'$.

Definition 4.17 Let $\{(F_A)_n\}$ be a sequence of fuzzy soft numbers in a fuzzy soft metric space $(U, E, \tilde{d})$. Then $\{(F_A)_n\}$ is said to be bounded, there exists a positive number $\beta\tilde{\in}(0, 1]$ such that $\tilde{d}((F_A)_n, (F_A)_m)\tilde{\leq}\beta, \forall n, m\tilde{\in}\beta$.

Definition 4.18 Let $\{(F_A)_n\}$ be a sequence of fuzzy soft numbers in a fuzzy soft metric space $(U, E, \tilde{d})$. Then $\{(F_A)_n\}$ is said to be Cauchy sequence of fuzzy soft numbers if for a positive number $\varepsilon\tilde{>}0$ there exists a positive integer β, such that

$$\tilde{d}((F_A)_n, (F_A)_m)\tilde{\leq}\varepsilon, \forall n, m\tilde{\geq}\beta\ i.e.\ \tilde{d}((F_A)_n, (F_A)_m)\tilde{\to}0 \text{ as } n, m\tilde{\to}\infty.$$

Theorem 4.19 Every convergent sequence in a fuzzy soft numbers is Cauchy sequence and every Cauchy sequence of fuzzy soft numbers is bounded.

Proof. Let $\{(F_A)_n\}$ be a sequence of fuzzy soft numbers in a fuzzy soft metric space $(U, E, \tilde{d})$.Let $\{(F_A)_n\}$ converges to $F_A^{'}$.For every $\varepsilon\tilde{>}0$ there exists a positive integer β such that

$$\tilde{d}((F_A)_n, F_A^{'})\tilde{\leq}\frac{\varepsilon}{2},\ \forall n \geq \beta.$$

Then for all $m, n \geq \beta$, we have

$$\tilde{d}((F_A)_n, (F_A)_m)\tilde{\leq}\tilde{d}((F_A)_n, F_A^{'}) + \tilde{d}(F_A^{'}, (F_A)_m)\tilde{<}\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence $\{(F_A)_n\}$ is a Cauchy sequence.
Let $\{(F_A)_n\}$ is a Cauchy sequence of fuzzy soft number. Therefore

$$\tilde{d}((F_A)_n, (F_A)_m)\tilde{\leq}\varepsilon\tilde{\in}(0, 1]$$

Hence $\{(F_A)_n\}$ is bounded.

Definition 4.20 A fuzzy soft numbers metric space $(U, E, \tilde{d})$ is called complete fuzzy soft metric space if every Cauchy sequence of fuzzy soft numbers in (U, E) converges to some fuzzy soft number in (U, E).

Definition 4.21 Let $\overline{(U, E)}$ and $\overline{(V, E^{'})}$ be classes of fuzzy soft numbers over U and V with attributes from E and $E^{'}$, respectively. Let $p : U \longrightarrow V$ uniformly one-one onto and $q : E \longrightarrow E^{'}$ be any mapping. Then $f = (p, q) : (U, E) \longrightarrow (V, E^{'})$ is a fuzzy soft numbers mapping.

Example 4.22 Let $U = \{h_1, h_2, h_3, h_4, h_5\}$ and $V = \{k_1, k_2, k_3, k_4, k_5\}, E = \{e_1, e_2, e_3\}, E' = \{e'_1, e'_2\}$ and $\overline{(U, E)}, \overline{(V, E')}$ classes of fuzzy soft numbers. Let $p(h_1) = k_1, p(h_2) = k_2, p(h_3) = k_3, p(h_4) = k_4, p(h_5) = k_5$ and $q(e_1) = e'_2, q(e_2) = e'_1, q(e_3) = e'_2$. Let us consider a fuzzy soft number H_A in $\overline{(U, E)}$ as

$$\begin{aligned} H_A = \{H(e_1) &= \{(h_1, 0.0), (h_2, 0.6), (h_3, 1.0), (h_4, 0.8), (h_5, 0.1)\}, \\ H(e_2) &= \{(h_1, 0.1), (h_2, 0.7), (h_3, 1.0), (h_4, 0.8), (h_5, 0.0)\}, \\ H(e_3) &= \{(h_1, 0.3), (h_2, 0.9), (h_3, 1.0), (h_4, 0.7), (h_5, 0.2)\}\}. \end{aligned}$$

Then the fuzzy soft number image of H_A under $f = (p, q) : \overline{(U, E)} \longrightarrow \overline{(V, E')}$ is obtained as

$$\begin{aligned} f(H_A)(e'_1)(k_1) &= \bigcup_{\alpha\tilde{\in}q^{-1}(e'_1)\cap A, s\tilde{\in}p^{-1}(k_1)} (\alpha)\mu_s \\ &= \bigcup_{\alpha\tilde{\in}\{e_2\}, s\tilde{\in}\{h_1\}} (\alpha)\mu_s \\ &= (e_2)\mu_{h_1} = \{0.1\} \end{aligned}$$

$$\begin{aligned} f(H_A)(e'_1)(k_2) &= \bigcup_{\alpha\tilde{\in}q^{-1}(e'_1)\cap A, s\tilde{\in}p^{-1}(k_2)} (\alpha)\mu_s \\ &= \bigcup_{\alpha\tilde{\in}\{e_2\}, s\tilde{\in}\{h_2\}} (\alpha)\mu_s \\ &= (e_2)\mu_{h_2} = \{0.7\}. \end{aligned}$$

By similar calculations we get

$$\begin{aligned} f(H_A) = \{e'_1 &= \{(k_1, 0.1), (k_2, 0.7), (k_3, 1.0), (k_4, 0.8), (k_5, 0.0)\}, \\ e'_2 &= \{(k_1, 0.3), (k_2, 0.9), (k_3, 1.0), (k_4, 0.8), (k_5, 0.2)\}\}. \end{aligned}$$

Again consider a fuzzy soft number H'_B in $\overline{(V, E')}$ as

$$\begin{aligned} H'_B = \{e'_1 &= \{(k_1, 0.2), (k_2, 0.8), (k_3, 1.0), (k_4, 0.7), (k_5, 0.1)\}, \\ e'_2 &= \{(k_1, 0.1), (k_2, 0.6), (k_3, 1.0), (k_4, 0.7), (k_5, 0.0)\}. \end{aligned}$$

Therefore

$$f^{-1}(H'_B)(e_1)(h_1) = (q(e_1))\mu_{p(h_1)} = (e'_2)\mu_{K_1} = \{0.1\}.$$

By similar calculations, we get

$$\begin{aligned} f^{-1}(H'_B) = \{e_1 &= \{(h_1, 0.1), (h_2, 0.6), (h_3, 1.0), (h_4, 0.7), (h_5, 0.0)\} \\ e_2 &= \{(h_1, 0.2), (h_2, 0.8), (h_3, 1.0), (h_4, 0.8), (h_5, 0.0)\} \\ e_3 &= \{(h_1, 0.1), (h_2, 0.6), (h_3, 1.0), (h_4, 0.7), (h_5, 0.0)\}\}. \end{aligned}$$

Proposition 4.23 Let $\overline{(U,E)}$ and $\overline{(V,E')}$ be classes of fuzzy soft numbers. Let $p : U \longrightarrow V$ be not uniformly one-one onto and $q : E \longrightarrow E^{'}$ be mapping. Then $f = (p, q) : (U, E) \longrightarrow (V, E^{'})$ is not a fuzzy soft numbers mapping.

Proof. From Example 4.22, here we consider $p(h_1) = k_3, p(h_2) = k_5, p(h_3) = k_1, p(h_4) = k_2, p(h_5) = k_4$. Then the image of H_A under $f = (p, q) : \overline{(U,E)} \longrightarrow \overline{(V,E')}$ is obtained as

$$\begin{aligned} f(H_A)(e'_1)(k_1) &= \bigcup_{\alpha\tilde{\in}q^{-1}(e'_1)\cap A, s\tilde{\in}p^{-1}(k_1)} (\alpha)\mu_s \\ &= \bigcup_{\alpha\tilde{\in}\{e_2\}, s\tilde{\in}\{h_3\}} (\alpha)\mu_s \\ &= (e_2)\mu_{h_3} = \{1.0\}. \end{aligned}$$

Similar we get

$$\begin{aligned} f(H_A) = \{e'_1 &= \{(k_1, 1.0), (k_2, 0.8), (k_3, 0.1), (k_4, 0.0), (k_5, 0.7)\}, \\ e'_2 &= \{(k_1, 1.0), (k_2, 0.8), (k_3, 0.3), (k_4, 0.2), (k_5, 0.9)\} \end{aligned}$$

which is not a fuzzy soft numbers. This complete the proof.

Definition 4.24 A fuzzy soft number mapping $f = (p, q) : (U, E) \longrightarrow (V, E^{'})$ is said to be a one-one and onto if $q : E \longrightarrow E^{'}$ be a one-one onto.

Theorem 4.25 Suppose a fuzzy soft number mapping $f = (p, q) : (U, E) \longrightarrow (V, E^{'})$ is one-one onto of two fuzzy soft metric spaces $(U, E, \tilde{d_1})$ and $(V, E^{'}, \tilde{d_2})$. If F_A and G_A are two fuzzy soft numbers of (U, E), then $\tilde{d_1}(F_A, G_A) = \tilde{d_2}(f(F_A), f(G_A))$.

Proof. Since f is one-one onto. Therefore p and q are also one-one onto. Therefore

$$\mu_{F(\cap_i(e_i))}(h_t) = \mu_{f(F)(\cap_i(e_i))}(h_t)$$

and

$$\mu_{G(\cap_i(e_i))}(h_t) = \mu_{f(G)(\cap_i(e_i))}(h_t)$$

for all $F_A, G_A\tilde{\in}(U, E)$.
Hence, $\tilde{d_1}(F_A, G_A) = \tilde{d_2}(f(F_A), f(G_A))$.

Definition 4.26 Let $(U, E, \tilde{d_1})$ and $(V, E^{'}, \tilde{d_2})$ be any two fuzzy soft metric spaces. A fuzzy soft number function (or mapping) $f : (U, E) \longrightarrow (V, E^{'})$ is said to be fuzzy soft continuous at $F^{'}_A$ of (U, E), if for given $\varepsilon\tilde{>}0$ there exists a $\delta\tilde{>}0$, such that $\tilde{d_2}(f(F_A), f(F^{'}_A))\tilde{\leq}\varepsilon$, whenever $\tilde{d_1}(F_A, F^{'}_A)\tilde{\leq}\delta$.
i.e., for each fuzzy soft open sphere $S_\epsilon(f(F^{'}_A))$ centered at $f(F^{'}_A)$ there is a fuzzy soft open sphere $S_\delta(F^{'}_A)$ centered at $F^{'}_A$ such that

$$f(S_\delta(F^{'}_A))\tilde{\subseteq}S_\varepsilon(f(F^{'}_A)).$$

Definition 4.27 A fuzzy soft number function $f : (U, E, \tilde{d_1}) \longrightarrow (V, E^{'}, \tilde{d_2})$ is said to be fuzzy soft continuous, if it is fuzzy soft continuous at each fuzzy soft number of (U, E).

Proposition 4.28 Every inverse image of soft number function is fuzzy soft continuous.

Proof. Proof of the results are obvious.

Theorem 4.29 Let $(U, E, \tilde{d_1})$ and $(V, E^{'}, \tilde{d_2})$ be any two fuzzy soft metric spaces and $f : (U, E) \longrightarrow (V, E^{'})$ be fuzzy soft continuous. Then for every soft number sequence $(F_A)_n$ converges to $F^{'}_A$ we have $\lim_{n\tilde{\to}\infty} f((F_A)_n) = f(F^{'}_A)$ i.e $(F_A)_n\tilde{\to}F^{'}_A \Rightarrow f((F_A)_n)\tilde{\to}f(F^{'}_A)$.

Proof. Since f is a fuzzy soft number function. Therefore f is a fuzzy soft continuous. Let f be a fuzzy soft continuous at $F_A^{'}$. Therefore for any given $\varepsilon\tilde{>}0$ there exists a $\delta\tilde{>}0$, such that

$$\tilde{d_2}(f(X_A), f(F_A^{'}))\tilde{\leq}\varepsilon, \text{ whenever } \tilde{d_1}(X_A, F_A^{'})\tilde{\leq}\delta. \quad (1)$$

Let us suppose that $(F_A)_n$ be a soft number sequence in $(U, E, \tilde{d_1})$, such that

$$\lim_{n\tilde{\to}\infty} f((F_A)_n) = f(F_A^{'}).$$

Since $\lim_{n\tilde{\to}\infty}(F_A)_n = F_A^{'}$, therefore there exists a positive integer m such that

$$\tilde{d_1}((F_A)_n, F_A^{'})\tilde{\leq}\delta, \forall n \geq m.$$

Now from (1), we get

$$\tilde{d_2}(f((F_A)_n), f(F_A^{'}))\tilde{\leq}\varepsilon, \forall n \geq m.$$

This implies that $\lim_{n\tilde{\to}\infty} f((F_A)_n) = f(F_A^{'})$.

Definition 4.30 Let $(U, E, \tilde{d_1})$ and $(V, E^{'}, \tilde{d_2})$ be any two fuzzy soft metric spaces. A fuzzy soft number function $f : (U, E) \longrightarrow (V, E^{'})$ is said to be fuzzy soft uniformly continuous if for each $\varepsilon\tilde{>}0$ there exists a $\delta\tilde{>}0$, such that

$$\tilde{d_2}(f(X_A), f(Y_A))\tilde{\leq}\varepsilon, \text{ whenever } \tilde{d_1}(X_A, Y_A)\tilde{\leq}\delta, \forall X_A, Y_A\tilde{\in}(U, E).$$

Proposition 4.31 Every fuzzy soft number function is uniformly continuous.

Proof. Proof of the results are obvious.

Theorem 4.32 Every fuzzy soft number function $f : (U, E) \longrightarrow (V, E^{'})$ which is uniformly continuous on (U, E) is necessarily soft continuous on (U, E).

Proof. Proof of the results are obvious.

Theorem 4.33 The continuous image of a fuzzy soft Cauchy sequence is again a fuzzy soft Cauchy sequence.

Proof. Let $(U, E, \tilde{d_1})$ and $(V, E^{'}, \tilde{d_2})$ be any two fuzzy soft metric spaces and $f : (U, E) \longrightarrow (V, E^{'})$ be fuzzy soft number function.

We have every soft number function is uniformly continuous. Therefore f is uniformly continuous.

Let $\{(F_A)_n\}$ be a fuzzy soft number Cauchy sequence in (U, E) and given $\varepsilon\tilde{>}0$. Then f being a fuzzy soft uniformly continuous, there exists a $\delta\tilde{>}0$ such that

$$\tilde{d_2}(f((F_A)_m), f((Y_A)_n))\tilde{\leq}\varepsilon \text{ whenever } \tilde{d_1}((F_A)_m, (F_A)_n)\tilde{\leq}\delta. \quad (2)$$

Since $\{(F_A)_n\}$ is fuzzy soft Cauchy sequence, corresponding to this $\delta\tilde{>}0$ there exists a positive integer n_0 such that

$$\tilde{d_1}((F_A)_m, (F_A)_n)\tilde{\leq}\delta \text{ for all } m, n\tilde{\geq}n_0. \quad (3)$$

From (2) and (3) we have

$$\tilde{d_2}(f((F_A)_m), f((Y_A)_n))\tilde{\leq}\varepsilon \text{ for all } m, n\tilde{\geq}n_0.$$

Hence $\{(F_A)_n\}$ is a soft Cauchy sequence in $(V, E^{'})$.

Definition 4.34 Let $(U, E, \tilde{d_1})$ and $(V, E^{'}, \tilde{d_2})$ be any two fuzzy soft metric spaces. A fuzzy soft number function $f = (p, q) : (U, E) \longrightarrow (V, E^{'})$ is said to be fuzzy soft homeomorphism if $q : E \longrightarrow E^{'}$ is one-one and onto.

Theorem 4.35 Every fuzzy soft number one one onto function is fuzzy soft homeomorphism.

Proof. Suppose $f = (p, q) : (U, E) \longrightarrow (V, E^{'})$ is one-one onto function. Therefore $p : U \longrightarrow V$ uniformly one-one onto and $q : E \longrightarrow E^{'}$ is one-one onto. Hence f is fuzzy soft homeomorphism.

Definition 4.36 Let $(U, E, \tilde{d_1})$ and $(V, E^{'}, \tilde{d_2})$ be any two fuzzy soft metric spaces. A fuzzy soft number function $f : (U, E) \longrightarrow (V, E^{'})$ is called a fuzzy soft isometry if

$$\tilde{d_1}(X_A, Y_A) = \tilde{d_2}(f(X_A), f(Y_A)), \forall X_A, Y_A\tilde{\in}(U, E).$$

Theorem 4.37 Every one one onto fuzzy soft number function is a fuzzy soft isometry function.

Proof. Follows from Theorem 4.35.

Theorem 4.38 Every fuzzy soft isometry function is not a one one onto fuzzy soft number function.

Proof. Consider the Example 4.22 and let

$$Q_A = \{Q(e_1) = \{(h_1, 0.3), (h_2, 0.9), (h_3, 1.0), (h_4, 0.7), (h_5, 0.2)\},$$
$$Q(e_2) = \{(h_1, 0.1), (h_2, 0.7), (h_3, 1.0), (h_4, 0.8), (h_5, 0.0)\},$$
$$Q(e_3) = \{(h_1, 0.0), (h_2, 0.8), (h_3, 1.0), (h_4, 0.7), (h_5, 0.0)\}\}.$$

Therefore

$$f(Q_A) = \{e'_1 = \{(k_1, 0.1), (k_2, 0.7), (k_3, 1.0), (k_4, 0.8), (k_5, 0.0)\},$$
$$e'_2 = \{(k_1, 0.3), (k_2, 0.9), (k_3, 1.0), (k_4, 0.7), (k_5, 0.2)\}\}.$$

Now

$$\tilde{d}_1(H_A, Q_A) = 0.1 = \tilde{d}_2(f(H_A), f(Q_A)).$$

So f is fuzzy soft isometry function but it is not a one one onto fuzzy soft number function.

Theorem 4.39 Every fuzzy soft isometry function is not a fuzzy soft homeomorphism.

Proof. From Theorem 4.38, every fuzzy soft isometry function is not a one one onto fuzzy soft number function. Therefore from definition of homeomorphism,every fuzzy soft isometry function is not a fuzzy soft homeomorphism.

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