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Fixed point approximation for generalized Chatterjea type contractive mappings in hyperbolic b-metric spaces

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Abstract: In this paper, we consider a generalized Chatterjea type contractive mapping and introduce an iteration scheme in a uniformly convex hyperbolic b-metric space. Strong convergence as well as Δ-convergence results for the introduced iteration scheme are obtained.

Keywords: Generalized Chatterjea type contractive mapping; iteration scheme; strong convergence; Δ-convergence.

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1 Introduction

The notion of a b-metric [2] is a generalization of a metric in which, for some $s \geq 1$, the triangle inequality is replaced by the b-triangle inequality:

$$\rho(x,y) \leq s\big[\rho(x,z) + \rho(z,y)\big].$$

As noted in [17], a b-metric need not be continuous.

If f is a self mapping on a b-metric space (X,d,s), then $F(f)$ denotes the set of all fixed points of f, i.e., $F(f) = \big\{x \in X : fx = x\big\}$. Let C be a closed subset of X and $f : C \longrightarrow C$ be a self mapping with $F(f) \neq \emptyset$. For $x_0 \in C$, consider the sequence of iterates $\{x_n\}$ given by

$$x_n = fx_{n-1} = f^n x_0, \qquad n = 1, 2, \ldots \tag{1}$$

The sequence defined by (1) is known as the sequence of successive approximation or simply Picard iteration. There is a natural interest in finding conditions on f, C and X, as general as possible, and which also guarantee the strong convergence of the sequence of iterates $\{x_n\}$ to a fixed point of f in C.

Moreover, if the sequence of iterates converges to a fixed point of f, it is interesting to evaluate the rate of convergence (or, alternately, the error estimate) of the method, i.e., in obtaining a stopping criterion for the sequence of successive approximation. For a weaker contractive condition, the Picard iterates need not converge to the fixed point of f, and some other iteration schemes must be considered. In this regard, many authors have introduced and investigated various iteration schemes to approximate fixed point for different classes of contractive conditions (for instance, refer to Agarwal et al. [1], Ishikawa [8], Krasnosel'skiĭ[12], Mann [16], Noor [18], Kadioglu & Yildirim [9], etc.).

In 2005, Kohlenbach [13] introduced hyperbolic spaces which can be extended to a hyperbolic b-space or hyperbolic b-metric space.

Definition 1.1. *[13] A hyperbolic b-space $(X, \mathcal{W}, d)$ is a b-metric space (X, d, s) such that there exists a convexity mapping $\mathcal{W} : X^2 \times [0,1] \longrightarrow X$ satisfying*

($\mathcal{W}_1$) $\rho\big(\mathcal{W}(x,y,\alpha), w\big) \leq (1-\alpha)\rho(x,w) + \alpha\rho(y,w)$

($\mathcal{W}_2$) $\rho\big(\mathcal{W}(x,y,\alpha), \mathcal{W}(x,y,\beta)\big) = |\alpha - \beta|\rho(x,y)$

($\mathcal{W}_3$) $\mathcal{W}(x,y,\alpha) = \mathcal{W}(x,y,1-\alpha)$

($\mathcal{W}_4$) $\rho\big(\mathcal{W}(x,v,\alpha),\mathcal{W}(y,w,\alpha)\big) \leq (1-\alpha)\rho(x,y)+\alpha\rho(v,w)$

for all v, w, x and y in X and $\alpha,\beta \in [0,1]$.

A convex b-metric space in the sense of Takahashi [20] is a b-metric space in which the triplet $(X,\mathcal{W},d)$ satisfy ($\mathcal{W}_1$). Examples of hyperbolic b-spaces includes normed linear spaces, the Hilbert ball with the hyperbolic metric, Cartesian products of Hilbert balls, Hadamard manifolds, CAT(0) spaces, etc. A more complete discussion on hyperbolic spaces and its examples can be found in [13].

Definition 1.2. *[14] A hyperbolic b-space $(X,\mathcal{W},d)$ is said to be uniformly convex if for any $r>0$ and $\varepsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that $\rho(x,a)\leq r$, $\rho(y,a)\leq r$ and $\rho(x,y)\geq \varepsilon r$ implies*

$$\rho\Big(\mathcal{W}\left(x,y,\tfrac{1}{2}\right),a\Big) \leq (1-\delta)r$$

for all $a,x,y \in X$.

A mapping $\eta:(0,\infty)\times(0,2]\longrightarrow(0,1]$ providing such a $\delta := \eta(r,\varepsilon)$ for a given $r>0$ and $\varepsilon\in(0,2]$ is called a *modulus of convexity*. η is said to be *monotone* if it decreases with r (for a fixed ε).

We now recall the concept of Δ-convergence and state its definitions in the setting of a b-metric space (one may also refer to [11]).

Definition 1.3. *[14] Let C be a nonempty subset of a b-metric space (X,d,s) and $\{x_n\}$ be a bounded sequence in X. Consider a continuous functional $r_a(\cdot\,,\{x_n\}):X\longrightarrow\mathbb{R}^+$ defined by*

$$r_a(x,\{x_n\}) = \limsup_{n\to\infty}\rho(x_n,x), \qquad x\in X.$$

The asymptotic radius of $\{x_n\}$ with respect to C is defined by

$$r_a\big(C,\{x_n\}\big)=\inf\Big\{r_a\big(x,\{x_n\}\big):x\in C\Big\},$$

and the asymptotic centre of the sequence $\{x_n\}$ with respect to C is defined by

$$A\big(C,\{x_n\}\big)=\Big\{x\in C: r_a\big(x,\{x_n\}\big)=r_a\big(C,\{x_n\}\big)\Big\}.$$

A point $z\in C$ is said to be an asymptotic centre of the sequence $\{x_n\}$ with respect to C if

$$r_a\big(z,\{x_n\}\big)=r_a\big(C,\{x_n\}\big)=\inf\Big\{r_a\big(x,\{x_n\}\big):x\in C\Big\}.$$

The set of all asymptotic centres of $\{x_n\}$ with respect to C, $A\big(C,\{x_n\}\big)$ could be empty, a singleton set or contains infinitely many points. If $C=X$, then $A\big(X,\{x_n\}\big)$ is simply denoted by $A\big(\{x_n\}\big)$. Similarly, $r_a\big(X,\{x_n\}\big)=r_a\big(\{x_n\}\big)$.

Definition 1.4. *[11] A sequence $\{x_n\}$ Δ-converges to x if $A(\{u_n\})=\{x\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$, that is, x is the unique asymptotic centre for every subsequence $\{u_n\}$ of $\{x_n\}$. It is denoted by $\Delta-\lim_{n\to\infty}x_n = x$.*

For $x\in X$, it is clear that $r_a\big(x,\{x_n\}\big)=0$ if and only if $\lim_{n\to\infty}x_n=x$. In CAT(0) spaces (and uniformly convex Banach spaces) it is known that for each closed convex subset K every bounded sequence has a unique $A\big(C,\{x_n\}\big)$ [3].

The following lemma is due to Leauştean [14] and this property also holds true for a uniformly convex hyperbolic b-metric space.

Lemma 1.5. *[14] Let $(X,\mathcal{W},d)$ be a uniformly convex hyperbolic b-metric space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic centre with respect to any nonempty closed convex subset C of X.*

In 2007, Agarwal et al. [1] introduced the S-iteration scheme which may be put in the following form for a hyperbolic b-metric space. Let C be a nonempty subset of a hyperbolic b-metric space $(X, \mathcal{W}, d)$. For $a_0 \in C$, define

$$\left.\begin{aligned} a_{n+1} &= \mathcal{W}(fa_n, fb_n, \alpha_n) \\ b_n &= \mathcal{W}(a_n, fa_n, \beta_n) \end{aligned}\right\} \tag{2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$.

In 2014, Kadioglu and Yildirim [9] defined Picard normal S-iteration scheme (refer [7]) for a convex subset of a normed space. The same iteration may be defined in a nonempty closed and convex subset C of a hyperbolic b-metric space as follows. For $r_0 \in C$,

$$\left.\begin{aligned} r_{n+1} &= fs_n \\ s_n &= \mathcal{W}(t_n, ft_n, \alpha_n) \\ t_n &= \mathcal{W}(r_n, fr_n, \beta_n) \end{aligned}\right\} \tag{3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$.

They showed that (3) converges faster than that of Picard, Mann, Ishikawa and (2).

Definition 1.6. *[7] Let C be a nonempty subset of a hyperbolic b-metric space $(X, \mathcal{W}, d)$ and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be* Fejér monotone *with respect to C if*

$$\rho(x_{n+1}, y) \leq \rho(x_n, y)$$

for all $y \in C$, $n \in \mathbb{N}$.

For $w \in F(f)$, we get from the above definition

$$\rho(x_{n+1}, w) \leq \rho(x_n, w) \qquad \forall\, n \in \mathbb{N}. \tag{4}$$

Since the sequence of positive real numbers $\{\rho(x_n, w)\}$ is monotonically decreasing, it must be convergent, say to $\mu \geq 0$. And therefore,

$$\lim_{m,n\to\infty} \rho(x_m, x_n) \leq 2s\mu,$$

from which we conclude $\{x_n\}$ is bounded. Hence we have the following lemma.

Lemma 1.7. *Let C be a nonempty subset of a hyperbolic b-metric space $(X, \mathcal{W}, d)$ and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ is Fejér monotone with respect to C, then*

1. *the sequence $\{\rho(x_n, w)\}$ is decreasing and hence convergent for all $w \in F(f)$,*
2. *$\{x_n\}$ is bounded.*

Lemma 1.8. *[10] Let $(X, \mathcal{W}, d)$ be a uniformly convex hyperbolic b-metric space with monotone modulus of uniform convexity η. Let $z \in X$ and $\{t_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$, $\{y_n\}$ are sequences in X such that* $\limsup_{n\to\infty} \rho(x_n, z) \leq c$, $\limsup_{n\to\infty} \rho(y_n, z) \leq c$ *and* $\limsup_{n\to\infty} \rho\big(\mathcal{W}(x_n, y_n, t_n), z\big) = c$ *for some $c \geq 0$, then*

$$\lim_{n\to\infty} \rho(x_n, y_n) = 0.$$

Definition 1.9. *A function $\phi : [0, \infty) \longrightarrow [0, \infty)$ is said to be a subadditive subhomogeneous altering distance function if*

(i) *ϕ is an altering distance function [4] (i.e., ϕ is continuous, strictly increasing and $\phi(t) = 0$ if and only if $t = 0$),*

(ii) $\phi(x + y) \leq \phi(x) + \phi(y), \qquad \forall\ x, y \in [0, \infty)$,

(iii) $\phi(\alpha x) \leq \alpha\phi(x)$, *for $\alpha \geq 1$.*

Examples of subadditive subhomogeneous altering distance functions are $\phi_1(x) = kx$ for some $k \geq 1$, $\phi_2(x) = \sqrt[n]{x},\ n \in \mathbb{N}$, $\phi_3(x) = \log(1+x),\ x \geq 0$, etc.

The following lemma is useful in the following discussions of fixed point existence results.

Lemma 1.10. *[15] Let (X, d, s) be a b-metric space and $\{x_n\}$ be a convergent sequence in X with $\lim_{n\to\infty} x_n = x$. Then for all $y \in X$*

$$s^{-1}\rho(x, y) \leq \lim_{n\to\infty} \inf \rho(x_n, y) \leq \lim_{n\to\infty} \sup \rho(x_n, y) \leq s\rho(x, y).$$

2 Main results

Using subadditive subhomogeneous altering distance functions, we define a generalized Chatterjea type contractive mapping as follows.

Definition 2.1. *Let (X, d, s) be a b-metric space. A mapping $f : X \longrightarrow X$ is said to be a generalized Chatterjea type contractive mapping if there exists $p < \frac{1}{s(s+1)}$ such that*

$$\phi\big(\rho(fx, fy)\big) \leq p\Big\{\phi\big(\rho(x, fy)\big) + \phi\big(\rho(y, fx)\big)\Big\} \tag{5}$$

for all $x, y \in X$.

Lemma 2.2. *Let f be a generalized Chatterjea type contractive mapping and $\{x_n\}$ be a sequence given by the Picard iteration $x_{n+1} = fx_n$, $n \geq 0$. Then $\{x_n\}$ is Fejér monotone with respect to $F = F(f)$.*

Proof. For $w \in F$, we have

$$\phi\big(\rho(fx, w)\big) \leq p\Big\{\phi\big(\rho(x, w)\big) + \phi\big(\rho(w, fx)\big)\Big\},$$

or,

$$\rho(fx, w) \leq k\rho(x, w)$$

where $k = \frac{p}{1-p} < 1$ and hence the result. □

Theorem 2.3. *Let (X, d, s) be a complete b-metric space with coefficient $s \geq 1$ and $f : X \longrightarrow X$ be a generalized Chatterjea type contractive mapping. Then f has a unique fixed point.*

Proof. Following [6] and using (5), we can show that $\{x_n\}$ is a Cauchy sequence. The completeness of X implies there exists $z \in X$ such that

$$\lim_{n\to\infty} x_n = z.$$

Now,

$$\begin{aligned}\phi\big(\rho(fz, z)\big) &\leq \phi\Big(s\rho(fz, fx_n) + s\rho(fx_n, z)\Big)\\ &\leq sp\Big\{\phi\big(\rho(z, fx_n)\big) + \phi\big(\rho(x_n, fz)\big)\Big\} + s\phi\big(\rho(x_{n+1}, z)\big)\\ &\leq sp\Big\{\phi\big(\rho(z, x_{n+1})\big) + \phi\big(\rho(x_n, fz)\big)\Big\} + s\phi\big(\rho(x_{n+1}, z)\big)\end{aligned}$$

By virtue of Lemma 1.10, taking the limit as $n \to \infty$, we get

$$(1 - sp)\phi\big(\rho(fz, z)\big) = 0,$$

and hence $\rho(fz, z) = 0$. The uniqueness of the fixed point follows from (5). □

Next, we prove the strong convergence as well as Δ-convergence of the iteration scheme generated by (6) to the fixed point of a generalized Chatterjea type contractive mapping, following [7] (One may also refer to [5]).

We define an iterative scheme for a self mapping f on a nonempty closed and convex subset C of a hyperbolic b-metric space $(X, \mathcal{W}, d)$ as follows. For $x_0 \in C$, define

$$\left.\begin{aligned} x_{n+1} &= \mathcal{W}\left(fz_n, fy_n, \alpha_n\right), \\ y_n &= \mathcal{W}\left(z_n, fz_n, \beta_n\right), \\ z_n &= fx_n \end{aligned}\right\} \tag{6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$.

Lemma 2.4. *Let C be a nonempty closed and convex subset of a hyperbolic b-metric space $(X, \mathcal{W}, d)$ and $f : C \longrightarrow C$ be a generalized Chatterjea type contractive mapping. If $\{x_n\}$ is a sequence generated by the iteration (6), then $\{x_n\}$ is Fejér monotone with respect to $F = F(f)$.*

Lemma 2.5. *Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic b-metric space $(X, \mathcal{W}, d)$ and $f : C \longrightarrow C$ be a generalized Chatterjea type contractive mapping. If $\{x_n\}$ is a sequence defined by the iteration scheme (6), then the sequence $\{x_n\}$ is bounded and*

$$\lim_{n\to\infty} \rho(x_n, fx_n) = 0.$$

Proof. Since $F(f) \neq \emptyset$, let $w \in F(f)$. Then by Lemma 2.4, $\{x_n\}$ is Fejér monotone with respect to $F = F(f)$ and by Lemma 1.7 $\{x_n\}$ is bounded and $\lim_{n\to\infty} \rho(x_n, w)$ exists. Let $\lim_{n\to\infty} \rho(x_n, w) = \mu \geq 0$. Now, for $w \in F$ we have

$$\phi\big(\rho(fx, w)\big) \leq p\Big\{\phi\big(\rho(x, w)\big) + \phi\big(\rho(w, fx)\big)\Big\},$$

from which we get

$$\rho(fx, w) \leq k\rho(x, w),$$

where $k = \frac{sp}{1-sp} < 1$.

If $\mu = 0$, then $\rho(x_n, fx_n) \leq 2s\rho(x_n, w)$ and taking the limit as $n \to \infty$, we get $\lim_{n\to\infty} \rho(x_n, fx_n) = 0$.

If $\mu > 0$, since $\rho(fx_n, w) \leq \rho(x_n, w)$, taking $\limsup$ as $n \to \infty$, we get

$$\limsup_{n\to\infty} \rho(fx_n, w) \leq \mu.$$

Now, since $y_n = \mathcal{W}(z_n, fz_n, \beta_n)$, we have

$$\rho(y_n, w) \leq (1 - \alpha_n)\rho(z_n, w) + \alpha_n\rho(fz_n, w) \leq \rho(z_n, w),$$

and as in the above, we get

$$\limsup_{n\to\infty} \rho(y_n, w) \leq \mu.$$

Since $\rho(x_{n+1}, w) \leq \rho(y_n, w)$, taking $\liminf$ as $n \to \infty$, we get

$$\mu = \liminf_{n\to\infty} \rho(x_{n+1}, w) \leq \liminf_{n\to\infty} \rho(y_n, w) \leq \mu,$$

i.e.,

$$\lim_{n\to\infty} \rho(y_n, w) = \mu.$$

This implies that

$$\begin{aligned} \mu = \limsup_{n\to\infty} \rho(y_n, w) &= \limsup_{n\to\infty} \rho\big(\mathcal{W}(z_n, fz_n, \beta_n)\big) \\ &\leq \limsup_{n\to\infty} \Big\{(1 - \beta_n)\rho(z_n, w) + \beta_n\rho(fz_n, w)\Big\} \\ &\leq \limsup_{n\to\infty} \Big\{(1 - \beta_n)\rho(x_n, w) + \beta_n\rho(fx_n, w)\Big\} \\ &\leq \limsup_{n\to\infty} \rho\big(x_n, w\big) = \mu, \end{aligned}$$

i.e.,

$$\limsup_{n\to\infty} \mathcal{W}(x_n, fx_n, \beta_n) = \mu.$$

It then follows from Lemma 1.8 that $\lim_{n\to\infty} \rho(x_n, fx_n) = 0$.

□

Theorem 2.6. *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic b-metric space $(X, \mathcal{W}, d)$ with monotone modulus of uniform convexity η, and $f : C \longrightarrow C$ be a generalized Chatterjea type mapping. If $\{x_n\}$ is the sequence generated by (6), then $\{x_n\}$ Δ-converges to the fixed point of f.*

Proof. We note that $\{x_n\}$ is a bounded sequence, by Lemma 2.5, and therefore $\{x_n\}$ has a Δ-convergent subsequence. To show that $\{x_n\}$ Δ-converges to the fixed point of f, we shall show that every Δ-convergent subsequence of $\{x_n\}$ has a unique Δ-limit in $F(f)$. To see this, let v and w be Δ-limits of the subsequences $\{v_n\}$ and $\{w_n\}$ of $\{x_n\}$, respectively. By Lemma 1.5, $A\big(C, \{v_n\}\big) = \{v\}$ and $A\big(C, \{w_n\}\big) = \{w\}$.

By Theorem 2.3, v and w are fixed points of f. The fact that $v = w$ follows as in the proof of Theorem 3.1 [7] and hence the result. □

Theorem 2.7. *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic b-metric space $(X, \mathcal{W}, d)$ with monotone modulus of uniform convexity η, and $f : C \longrightarrow C$ be a generalized Chatterjea type mapping. Then the sequence $\{x_n\}$ generated by (6) converges strongly to the fixed point of f if and only if $\liminf_{n\to\infty} D(x_n, F(f)) = 0$, where $D(x_n, F(f)) = \inf_{x\in F(f)} \rho(x_n, x)$.*

Proof. If $\{x_n\}$ defined by (6) strongly converges to a fixed point of f, then obviously $\liminf_{n\to\infty} D(x_n, F(f)) = 0$.

To show the sufficiency part, we first note that $F(f)$ is closed. Now, from (4) we get

$$D(x_{n+1}, F(f)) \le D(x_n, F(f)),$$

which by Lemma 1.7 and Lemma 2.2 implies that $\lim_{n\to\infty} \rho(x_n, F(f))$ exists and since $\liminf_{n\to\infty} \rho(x_n, F(f)) = 0$, we get that

$$\lim_{n\to\infty} \rho(x_n, F(f)) = 0.$$

Consider a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\rho(x_{n_k}, w_k) < \frac{1}{2^k}$ for all $k \ge 1$ and $\{w_k\} \subseteq F(f)$. By Lemma 2.2

$$\rho(x_{n_{k+1}}, w_k) \le \rho(x_{n_k}, w_k) < \frac{1}{2^k}$$

which implies

$$\rho(w_{k+1}, w_k) \le s\Big(\rho(w_{k+1}, x_{n_{k+1}}) + \rho(x_{n_{k+1}}, w_k)\Big) < \frac{s}{2^{k-1}},$$

showing that $\{w_k\}$ is a Cauchy sequence. Since $F(f)$ is closed, $\{w_k\}$ converges in $F(f)$. Let $\lim_{n\to\infty} w_k = w$. Then

$$\rho(x_{n_k}, w) \le s\rho(x_{n_k}, w_k) + s\rho(w_k, w) \longrightarrow 0 \quad \text{as } \ k \to \infty$$

implies $\lim_{k\to\infty} \rho(x_{n_k}, w) = 0$ and since $\lim_{n\to\infty} \rho(x_n, w)$ exists, we have

$$\lim_{n\to\infty} \rho(x_n, w) = 0,$$

as required. □

Next, we prove a strong convergence result using the definition of condition (*I*) given by Senter and Doston [19] for metric spaces. The definition can be extended to b-metric spaces as well.

Definition 2.8. *[19] Let C be a nonempty subset of a b-metric spaces (X, d, s). A mapping $f : C \longrightarrow C$ with $F(f) \ne \emptyset$ is said to satisfy Condition (I) if there exists a non-decreasing function $\beta : [0,\infty) \longrightarrow [0,\infty)$ with $\beta(0) = 0$, $\beta(t) > 0$ for all $t \in (0,\infty)$ such that*

$$\beta\left(D(x, F(f))\right) \le \rho(x, fx) \quad \textit{for all } \ x \in C,$$

where $D(x, F(f)) = \inf\big\{\rho(x, w) : w \in F(f)\big\}$.

Theorem 2.9. *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic b-metric space $(X, \mathcal{W}, d)$ with monotone modulus of uniform convexity η, and $f : C \longrightarrow C$ be a generalized Chatterjea type mapping. If f satisfy Condition (I), then the sequence defined by (6) converges strongly to the fixed point of f.*

Proof. As in the proof of Theorem 2.7, $F(f)$ is closed. We observe that by Lemma 2.5, $\lim_{n\to\infty} \rho(x_n, fx_n) = 0$. Since f satisfy Condition (I), we have

$$\lim_{n\to\infty} \beta\left(D(x_n, F(f))\right) \leq \lim_{n\to\infty} \rho(x_n, fx_n) = 0.$$

Since β is a non-decreasing function $\eta : [0, \infty) \longrightarrow [0, \infty)$ with $\beta(0) = 0$, $\beta(t) > 0$ for all $t \in (0, \infty)$,

$$\lim_{n\to\infty} \rho\big((x_n, F(f)\big) = 0.$$

The conclusion of the proof follows from that of Theorem 2.7. □

Corollary 2.10. *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic b-metric space $(X, \mathcal{W}, d)$ with monotone modulus of uniform convexity η, and $f : C \longrightarrow C$ be a generalized Chatterjea type mapping. If $\{x_n\}$ is the sequence generated by*

$$\left.\begin{aligned} x_{n+1} &= \mathcal{W}\left(fx_n, fy_n, \alpha_n\right), \\ y_n &= \mathcal{W}\left(x_n, fx_n, \beta_n\right), \end{aligned}\right\} \tag{7}$$

where $\{t_n\}$ and $\{\alpha_n\}$ are real sequences in $(0, 1)$, then $\{x_n\}$ Δ-converges to the fixed point of f.

Corollary 2.11. *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic b-metric space $(X, \mathcal{W}, d)$ with monotone modulus of uniform convexity η, and $f : C \longrightarrow C$ be a generalized Chatterjea type mapping. Then the sequence defined by (7) converges strongly to the fixed point of f if and only if $\liminf_{n\to\infty} D\left(x_n, F(f)\right) = 0$, where $D\left(x_n, F(f)\right) = \inf_{x\in F(f)} \rho(x_n, x)$.*

Corollary 2.12. *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic b-metric space $(X, \mathcal{W}, d)$ with monotone modulus of uniform convexity η, and $f : C \longrightarrow C$ be a generalized Chatterjea type mapping. If f satisfy Condition (I), then the sequence defined by (7) converges strongly to some fixed point of f.*

Conclusion

In this paper, we considered a generalized Chatterjea type contractive mapping and obtained a fixed point result. An iteration scheme is introduced and it's strong convergence as well as Δ-convergence for the introduced mappings in a complete uniformly convex hyperbolic b-metric space are obtained. The rate of convergence (to the fixed point) of the introduced iteration may also be compared with known iteration schemes.

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