On ideal-ward compactness

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Abstract: A family of sets $I \subset 2^{\mathbf{N}}$ is called an ideal if and only if for each $A, B \in I$, implies $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$, implies $B \in I$. A real function f is ward continuous if and only if $(\Delta f(x_n))$ is a null sequence whenever (x_n) is a null sequence and a subset E of **R** is ward compact if any sequence $\mathbf{x} = (x_n)$ of points in E has a quasi-Cauchy subsequence where **R** is the set of real numbers. These recent known results suggest to us introducing a concept of I-ward continuity in the sense that a function f is I-ward continuous if $I - \lim_{n\to\infty} \Delta f(x_n) = 0$ whenever $I - \lim_{n\to\infty} \Delta x_n = 0$ and a concept of I-ward compactness in the sense that a subset E of **R** is I-ward compact if any sequence $\mathbf{x} = (x_n)$ of points in E has a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence **x** such that $I - \lim_{k\to\infty} \Delta z_k = 0$ where $\Delta z_k = z_{k+1} - z_k$. We investigate I-ward continuity and I-ward compactness, and prove some related problems

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1 Introduction

The concept of ideal convergence as a generalization of statistical convergence, and any concept involving statistical convergence play a vital role not only in pure mathematics but also in other branches of mathematics especially in information theory, computer science, and biological science.

A subset E of **R** is compact if any open covering of E has a finite subcovering where **R** is the set of real numbers. This is equivalent to the statement that any sequence $\mathbf{x} = (x_n)$ of points in E has a convergent subsequence whose limit is in E. A real function f is continuous if and only if $(f(x_n))$ is a convergent sequence whenever (x_n) is convergent. Regardless of limit, this is equivalent to the statement that $(f(x_n))$ is a Cauchy sequence whenever (x_n) is. Using the idea of continuity of a real function and the idea of compactness in terms of sequences, Çakallı [1] introduced the concept of forward continuity in the sense that a function f is forward continuous if it transforms forward convergent to 0 sequences to forward convergent to 0 sequences, i.e. $(f(x_n))$ is forward convergent to 0 whenever (x_n) is forward convergent to 0, and introduced the concept of forward compactness in the sense that a subset E of **R** is forward compact if any sequence $\mathbf{x} = (x_n)$ of points in E has a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence **x** such that $\lim_{k\to\infty} \Delta z_k = 0$ where $\Delta z_k = z_{k+1} - z_k$ (see also [2] where those sequences were called as quasi-Cauchy).

The purpose of this paper is to investigate the concept of ideal ward continuity of a real function and ideal ward compactness of subset of **R** and prove interesting theorems.

2 Preliminaries

First of all, some definitions and notation will be given in the following. Throughout this paper, **N** will denote the set of all positive integers. We will use boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ... for sequences $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n)$, ... of terms in **R**. c, S, I and ΔI will denote the set of all convergent sequences, statistically convergent sequences, I-convergent sequences and the set of all I-quasi-Cauchy sequences of points in **R** (again note that quasi-Cauchy sequences are forward convergent to zero sequences).

The idea of statistical convergence was formerly given under the name "almost convergence" by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [3]. The concept was formally introduced by Fast [4] and later was reintroduced by Schoenberg [5], and also independently by Buck [6].

The concept of statistical convergence is a generalization of the usual notion of convergence that, for real-valued sequences, parallels the usual theory of convergence. For a subset M of $\mathbf{N}$ the asymptotic density of M, denoted by $\delta(M)$, is given by

$$\delta(M) = \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : k \in M\}|,$$

if this limit exists, where $|\{k \leq n : k \in M\}|$ denotes the cardinality of the set $\{k \leq n : k \in M\}$. A sequence (x_n) is statistically convergent to ℓ if

$$\delta(\{n : |x_n - \ell| \geq \epsilon\}) = 0,$$

for every $\epsilon > 0$. In this case ℓ is called the statistical limit of $\mathbf{x}$. Schoenberg [5] studied some basic properties of statistical convergence and also studied the statistical convergence as a summability method. Fridy [7] gave characterizations of statistical convergence.

The notion of I-convergence initially introduced by Kostyrko et al. [8]. Let X be a non-empty set, then a family of sets $I \subset 2^X$ (the class of all subsets of X) is called an *ideal* if and only if for each $A, B \in I$, we have $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$, we have $B \in I$. A non-empty family of sets $F \subset 2^X$ is a *filter* on X if and only if $\Phi \notin F$, for each $A, B \in F$, we have $A \cap B \in F$ and each $A \in F$ and each $A \subset B$, we have $B \in F$. An ideal I is called *non-trivial* ideal if $I \neq \Phi$ and $X \notin I$. Clearly $I \subset 2^X$ is a non-trivial ideal if and only if $F = F(I) = \{X - A : A \in I\}$ is a filter on X. A non-trivial ideal $I \subset 2^X$ is called *admissible* if and only if $\{\{x\} : x \in X\} \subset I$. A non-trivial ideal I is maximal if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset. Further details on ideals of 2^X can be found in Kostyrko, et.al (see [8]).

Recall that a sequence $\mathbf{x} = (x_n)$ of points in $\mathbf{R}$ is said to be I- convergent to the number ℓ if for every $\varepsilon > 0$, the set $\{n \in \mathbf{N} : |x_n - \ell| \geq \varepsilon\} \in I$. In this case we write $I - \lim x_n = \ell$. We see that I-convergence of a sequence (x_n) implies that $I - \lim_{n\to\infty} \Delta x_n = 0$.

Connor and Grosse-Erdman [9] gave sequential definitions of continuity for real functions calling G-continuity instead of A-continuity (see [10]) and their results covers the earlier works related to A-continuity where a method of sequential convergence, or briefly a method, is a linear function G defined on a linear subspace of the space of all sequences s, denoted by c_G, into $\mathbf{R}$. A sequence $\mathbf{x} = (x_n)$ is said to be G-convergent to ℓ if $\mathbf{x} \in c_G$ and $G(\mathbf{x}) = \ell$. In particular, $\lim$ denotes the limit function $\lim \mathbf{x} = \lim_n x_n$ on the linear space c and $st - \lim$ denotes the statistical limit function $st - \lim \mathbf{x} = st - \lim_n x_n$ on the linear space $st(\mathbf{R})$. Also $I - \lim$ denotes the $I-$ limit function $I - \lim \mathbf{x} = I - \lim_n x_n$ on the linear space $I(\mathbf{R})$. A function f is called G-continuous at a point u provided that whenever a sequence $\mathbf{x} = (x_n)$ of terms in the domain of f is G-convergent to u, then the sequence $f(\mathbf{x}) = (f(x_n))$ is G-convergent to $f(u)$. A method G is called regular if every convergent sequence $\mathbf{x} = (x_n)$ is G-convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. A method is called subsequential if whenever $\mathbf{x}$ is G-convergent with $G(\mathbf{x}) = \ell$, then there is a subsequence (x_{n_k}) of $\mathbf{x}$ with $\lim_k x_{n_k} = \ell$. Since the ordinary convergence implies ideal convergence, so I is a regular sequential method. Recently, Cakalli gave new sequential definitions of compactness and slowly oscillating compactness in [11],[12], and [13].

3 I-ward continuity and I-ward compactness

Definition: A sequence $\mathbf{x} = (x_n)$ is I-ward convergent to a number ℓ if $I - \lim_{n\to\infty} \Delta x_n = \ell$ where $\Delta x_n = x_{n+1} - x_n$. For the special case $\ell = 0$ we say that $\mathbf{x}$ is ideal quasi-Cauchy, or I-quasi-Cauchy, in place of I-forward convergent to 0. Thus a sequence (x_n) of points of $\mathbf{R}$ is I-quasi-Cauchy if (Δx_n) is I-convergent to 0.

Definition: A subset E of $\mathbf{R}$ is called I-ward compact if whenever $\mathbf{x} = (x_n)$ is a sequence of points in E there is a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of $\mathbf{x}$ such that $I - \lim_{k\to\infty} \Delta z_k = 0$.

Firstly, we note that any finite subset of $\mathbf{R}$ is I-ward compact, union of two I-ward compact subsets of $\mathbf{R}$ is I-ward compact and intersection of any I-ward compact subsets of $\mathbf{R}$ is I-ward compact. Furthermore any subset of an I-ward compact set is I-ward compact and any bounded subset of **R** is I-ward compact. Any compact subset of **R** is also I-ward compact, and the set **N** is not I-ward compact. We note that any slowly oscillating compact subset of **R** is I-ward compact (see [13]) for the definition of slowly oscillating compactness).

We note that this definition of I-ward compactness can not be obtained by any G-sequential compactness, i.e. by any summability matrix A, even by the summability matrix $A = (a_{nk})$ defined by $a_{nk} = -1$ if $k = n$ and $a_{kn} = 1$ if $k = n + 1$ and

$$G(x) = I - \lim A\mathbf{x} = I - \lim_{k\to\infty} \sum_{n=1}^{\infty} a_{kn}x_n = I - \lim_{k\to\infty} \Delta x_k \qquad (1)$$

(see [11] for the definition of G-sequential compactness). Despite that G-sequential compact subsets of **R** should include the singleton set $\{0\}$, I-forward compact subsets of **R** do not have to include the singleton $\{0\}$.

We recall the definitions of G-sequentially compactness of a subset E of **R** and G-sequentially continuity of a real function f.

A subset E of **R** is called G-sequentially compact if whenever (x_n) is a sequence of points in E there is subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ of (x_n) whose $G(\mathbf{y}) = \lim \mathbf{y}$ in E (see [14]).

For regular methods any sequentially compact subset E of **R** is also G-sequentially compact and the converse is not always true.

For any regular subsequential method G, a subset E of **R** is G-sequentially compact if and only if it is sequentially compact in the ordinary sense.

Definition: A subset E of **R** is called I-sequentially compact if whenever (x_n) is a sequence of points in E there is I-convergent subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ of (x_n) such that $I - \lim \mathbf{y}$ is in E.

Definition: A function $f : E \to \mathbf{R}$ is I-sequentially continuous at a point x_0 if, given a sequence (x_n) of points in E, $I - \lim \mathbf{x} = x_0$ implies that $I - \lim f(\mathbf{x}) = f(x_0)$.

It immediately follows from the fact that any I-convergent sequence of points in X with a I-limit ℓ has a convergent subsequence with the same limit ℓ in the ordinary sense (see [15] and [16]) that the sequential method I is regular and subsequential.

A sequence (x_n) of points in **R** is called slowly oscillating if for any given $\varepsilon > 0$; there exists $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that $|x_m - x_n| < \varepsilon$ if $n \geq N(\varepsilon)$ and $n \leq m \leq (1 + \delta)n$ (see [11]).

A subset E of **R** is said to be slowly oscillating compact if a sequence $\mathbf{x} = (x_n)$ of points of E there is a slowly oscillating subsequence $\mathbf{z} = (z_k)$ of $\mathbf{x}$(see [11]).

Theorem 1. *A subset E of* **R** *is I-ward compact if and only if it is bounded.*

Proof. Suppose that E is an I-ward compact subset of **R**. To show that E is bounded. If possible suppose that E is unbounded. If it is unbounded above. Then one can construct a sequence (x_n) of terms in E such that $x_{n+1} > 1 + x_n$ for each positive integer n. Then the set of terms of the sequence (x_n) is not I-ward compact. Similarly we can prove that if E is bounded below. Conversely suppose that E is a bounded subset of **R**. To show that E is I-ward compact. For this we have to show that for any sequence (x_n) of points in E has a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence (x_n) such that $I - \lim_{k\to\infty} \Delta z_k = 0$. Since (x_n) is a sequence of points in E so (x_n) is also a sequence of points in $\overline{E}$, where $\overline{E}$ is the closure of E. As $\overline{E}$ is sequentially compact, so there is a convergent subsequence $(z_k) = (x_{n_k})$ of (x_n) (no matter the limits is in E or not). This subsequence is I-convergent since I is a regular method. Hence $I - \lim_{k\to\infty} \Delta z_k = 0$. This completes the proof of the theorem.

Corollary 2. *Let E be a subset of* **R**. *The following statements are equivalent:*

(i) E *is bounded.*

(ii) E *is* I*-ward compact.*

(iii) E *is ward compact.*

(iv) E *is slowly oscillating compact.*

Definition: A function f is called I-ward continuous on E if $I - \lim_{n\to\infty} \Delta f(x_n) = 0$ whenever $I - \lim_{n\to\infty} \Delta x_n = 0$, for a sequence $\mathbf{x} = (x_n)$ of terms in E.

Theorem 3. *If* f *is* I*-ward continuous on a subset* E *of* **R**, *then it is* I*-continuous on* E.

Proof. Suppose that f is an I-ward continuous function on a subset E of **R**. Let (x_n) be an I-convergent sequence with $I - \lim_{n\to\infty} x_n = x_0$. Then we have the sequence

$$(x_1, x_0, x_2, x_0, x_3, x_0, ..., x_{n-1}, x_0, x_n, x_0, ...)$$

such that $I - \lim_{n\to\infty} \Delta x_n = 0$. Since f is I-forward contonuous, then the sequence

$$(y_n) = (f(x_1), f(x_0), f(x_2), f(x_0), ..., f(x_n), f(x_0), ...)$$

is a I-quasi-Cauchy sequence. Theofore $I - \lim_{n\to\infty} \Delta y_n = 0$. Hence $I - \lim_{n\to\infty}[f(x_n) - f(x_0)] = 0$. It follows that the sequence $(f(x_n))$ I-converges to $f(x_0)$. This completes the proof of the theorem.

Remark: The converse of the above theorem is not always true, for example we can consider the function $f(x) = x^2$, since $I - \lim_{n\to\infty} \Delta x_n = 0$ for the sequence $(x_n) = (\sqrt{n})$. But $I - \lim_{n\to\infty} \Delta f(x_n) \neq 0$, because $(f(\sqrt{n})) = (n)$.

Corollary 4. *If* f *is* I*-ward continuous, then it is* I*-sequentially continuous.*

Theorem 5. *If* f *is* I*-ward continuous, then it is* G*-sequentially continuous for any regular subsequential method* G.

Proof. The proof is easy, so omitted.

Corollary 6. *If* f *is* I*-ward continuous, then it is continuous in the ordinary sense.*

Proof. The proof is straightforward, so is omitted.

Theorem 7. An I*-ward continuous image of any* I*-ward compact subset of* **R** *is* I*-ward compact.*

Proof. Suppose that f is an I-ward continuous function on a subset E of **R** and E is an I-ward compact subset of **R**. Let (y_n) be a sequence of points in $f(E)$. Write $y_n = f(x_n)$ where $x_n \in E$ for each $n \in$ **N**. I-ward compactness of E implies that there is a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of (x_n) with $I - \lim_{k\to\infty} \Delta z_k = 0$. Write $(t_k) = (f(z_k))$. As f is I-ward continuous, so we have $I - \lim_{k\to\infty} \Delta f(z_k) = 0$. Thus we have obtained a subsequenc (t_k) of the sequence $(f(x_n))$ with $I - \lim_{k\to\infty} \Delta t_k = 0$. Thus $f(E)$ is I-ward compact. This completes the proof of the theorem.

Corollary 8. *An* I*-ward continuous image of any compact subset of* **R** *is compact.*

Proof. The proof of this theorem follows from the preceding theorem.

Corollary 9. *An* I*-ward continuous image of any bounded subset of* **R** *is bounded.*

Proof. The proof follows from Theorem 1 and Theorem 7.

Corollary 10. *An I-ward continuous image of an I-sequentially compact subset of* $\mathbf{R}$ *is G-sequentially compact for any regular subsequential method G.*

Theorem 11 *If (f_n) is a sequence of I-ward continuous functions defined on a subset E of* $\mathbf{R}$ *and (f_n) is uniformly convergent to a function f, then f is I-ward continuous on E.*

Proof. Let $\varepsilon > 0$ and (x_n) be a sequence of points in E such that $I - \lim_{n\to\infty} \Delta x_n = 0$. By the uniform convergence of (f_n) there exists a positive integer N such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in E$ whenever $n \geq N$. By the definition of ideal for all $x \in E$, we have

$$\{n \in \mathbf{N} : |f_n(x) - f(x)| \geq \frac{\varepsilon}{3}\} \in I.$$

As f_N is I-ward continuous on E we have

$$\{n \in \mathbf{N} : |f_N(x_{n+1}) - f_N(x_n)| \geq \frac{\varepsilon}{3}\} \in I.$$

On the other hand we have

$$\{n \in \mathbf{N} : |f(x_{n+1}) - f(x_n)| \geq \frac{\varepsilon}{3}\} \subseteq \{\{n \in \mathbf{N} : |f(x_{n+1}) - f_N(x_{n+1})| \geq \frac{\varepsilon}{3}\}$$
$$\cup\{n \in \mathbf{N} : |f_N(x_{n+1}) - f_N(x_n)| \geq \frac{\varepsilon}{3}\} \cup \{n \in \mathbf{N} : |f_N(x_n) - f(x_n)| \geq \frac{\varepsilon}{3}\}\}. \quad (3)$$

Since I is an admissible ideal, so the right hand side of the relation (3) belongs to I, we have

$$\{n \in \mathbf{N} : |f(x_{n+1}) - f(x_n)| \geq \frac{\varepsilon}{3}\} \in I.$$

This completes the proof of the theorem.

Theorem 12 *The set of all I-ward continuous functions on a subset E of* $\mathbf{R}$ *is a closed subset of the set of all continuous functions on E, i.e. $\overline{\Delta iwc(E)} = \Delta iwc(E)$ where $\Delta iwc(E)$ is the set of all I-ward continuous functions on E, $\overline{\Delta iwc(E)}$ denotes the set of all cluster points of $\Delta iwc(E)$.*

Proof. Let f be an element in $\overline{\Delta iwc(E)}$. Then there exists sequence (f_n) of points in $\Delta iwc(E)$ such that $\lim_{n\to\infty} f_n = f$. To show that f is I-ward continuous consider a sequence (x_n) of points in E such that $I - \lim_{n\to\infty} \Delta x_n = 0$. Since (f_n) converges to f, there exists a positive integer N such that for all $x \in E$ and for all $n \geq N$, $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$. By the definition of ideal for all $x \in E$, we have

$$\{n \in \mathbf{N} : |f_n(x) - f(x)| \geq \frac{\varepsilon}{3}\} \in I.$$

As f_N is I-ward continuous on E we have

$$\{n \in \mathbf{N} : |f_N(x_{n+1}) - f_N(x_n)| \geq \frac{\varepsilon}{3}\} \in I.$$

On the other hand we have

$$\{n \in \mathbf{N} : |f(x_{n+1}) - f(x_n)| \geq \frac{\varepsilon}{3}\} \subseteq \{\{n \in \mathbf{N} : |f(x_{n+1}) - f_N(x_{n+1})| \geq \frac{\varepsilon}{3}\}$$
$$\cup\{n \in \mathbf{N} : |f_N(x_{n+1}) - f_N(x_n)| \geq \frac{\varepsilon}{3}\} \cup \{n \in \mathbf{N} : |f_N(x_n) - f(x_n)| \geq \frac{\varepsilon}{3}\}\}. \quad (4)$$

Since I is an admissible ideal, so the right hand side of the relation (4) belongs to I, we have

$$\{n \in \mathbf{N} : |f(x_{n+1}) - f(x_n)| \geq \frac{\varepsilon}{3}\} \in I.$$

This completes the proof of the theorem.

Corollary 13 *The set of all I-ward continuous functions on a subset E of* $\mathbf{R}$ *is a complete subspace of the space of all continuous functions on E.*

Proof. The proof follows from the preceding theorem.

4 Further problems

Finally we note the following further investigation problems arise.

Problem 1. For further study , we suggest to investigate I-quasi-Cauchy sequences of fuzzy points and I-ward continuity for the fuzzy functions. However due to the change in settings, the definitions and methods of proofs will not always be analogous to these of the present work(for example see [19]).

Problem 2. For another further study we suggest to introduce a new concept in dynamical sysytems using I-ward continuity.

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