INITIAL VALUE SEARCHING TECHNIQUE TO OBTAIN NUMERICALLY THE BIFURCATION PARAMETER AS WELL AS PERIODIC POINTS IN ONE DIMENSIONAL DISCRETE DYNAMICAL SYSTEM

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Abstract: In this paper we apply a suitable and efficient numerical technique to obtain the periodic solutions arising in period doubling scenario of one dimensional discrete dynamical system. Applying this technique we obtain bifurcation points upto period 2^{11} with good accuracy which is a satisfactory result to predict accumulation point and which is verified by Lyapunov exponent.

1. Introduction:

Many one dimensional discrete dynamical systems follow period doubling route to chaos. Hence it is often desired to obtain the bifurcation points to get a sufficient accuracy in predicting the start of chaos[6]. If we consider a one dimensional unimodal dynamical system containing only one parameter say $x_{n+1} = f(x_n, a), x_n \in R$ and $'a'$ is the control parameter, then the periodic points may be obtained [3,4]by the equation

$$f^n(x,a) = x \quad (1)$$

and the nth bifurcation parameter "a" may be obtained by solving the set of equations $f^n(x,a) = x$

$$f^{n'}(x,a) = -1 \quad (2)$$

Suitable analytical procedure may be applied to solve (2) for small values of n, however it is not always possible to solve (2) analytically for large values of n. Different types of numerical procedures like bisection method, Newton-Raphson method, Secant method etc may be applied to obtain the periodic solutions of (2).

Bisection method is a good technique however the procedure makes the result slower for large values of n as compared to Newton-Raphson method. Again the Newton-Raphson method needs a good initial value to achieve the goal.

In this paper we discuss a procedure to search a good initial value for obtaining the periodic solutions by Newton Raphson method.

2. Numerical Method:

First step is to solve the polynomial equation $f^n(x,a) = x$ for a particular value of a, for a known value of n. It is assumed that for a particular value of a the function has stable periodic points of period n. First we solve the equation $f(x,a) = x$. For this we partition the interval $[0, f(x_c, a)]$ (where x_c represents the maximum value of f) into 10 sub-intervals say $]x_i, x_{i+1}]$ where $x_{i+1} = x_i + h$ and $h = \frac{f(x_c, a)}{10}$. We assume that the system has a unique stable fixed point in the interval $[0, f(x_c, a)]$ for a certain range of control parameter a and which becomes unstable at the first bifurcation point. Since the function has only one fixed point , the fixed point will lie in exactly one of the subintervals and by Descartes rule for a particular interval say $[x_i, x_{i+1}]$ we shall have $(f(x_i, a) - x_i)(f(x_{i+1}, a) - x_{i+1}) < 0$.

In this way we may obtain the interval in which the solution lies and the length of the interval is ten times smaller than the original interval. The mid value of the end points may be a good initial value for applying Newton Raphson technique.Thus after some steps we may easily get the fixed points for a particular value of a. Let it be x_1.

It is easy to observe that the period two points are on either side of period one point i.e the two periodic points lie in the disjoint intervals $[0, x_1[$ and $]x_1, f(x_c, a)]$. We can consider only one of the subintervals $[0, x_1[$ where only one of the periodic points of period 2 lie. So partitioning the interval $[0, x_1[$ into 10 sub-intervals where ith subinterval will be $[x_i, x_i + h]$ and $h = \frac{x_1}{10}$. We may observe that exactly one of the subintervals will satisfy the relation $(f^2(x_i, a) - x_i)(f^2(x_i + h, a) - x_i - h) < 0$.

After identifying the suitable subinterval we may start Newton Raphson method to obtain the solution of period 2. Once we obtain the period two points say $x_{2,1}, x_{2,2}$ we may observe that exactly one periodic points of period 4 will lie in each of the sub-intervals $(o, x_{2,1}), (x_{2,1}, x_1), (x_1, x_{2,2}), (x_{2,2}, f(x_c))$. Hence again repeating the above procedure we may get the periodic solution of period 4 and so on.

3. Results:

Here we consider the difference equation[1] $x_{n+1} = (1-\gamma)\left(\alpha x_n + x_n e^{a\left\{1-\left(\frac{x_n}{k}\right)^{\theta}\right\}}\right)$ ……..(3) representing the population model having different types of species given by the parameter θ [5]with constant adult survivor rate given by α and γ is the constant effort harvesting strategy and a is the growth parameter[7,8,12]

The bifurcation diagram is as follows.

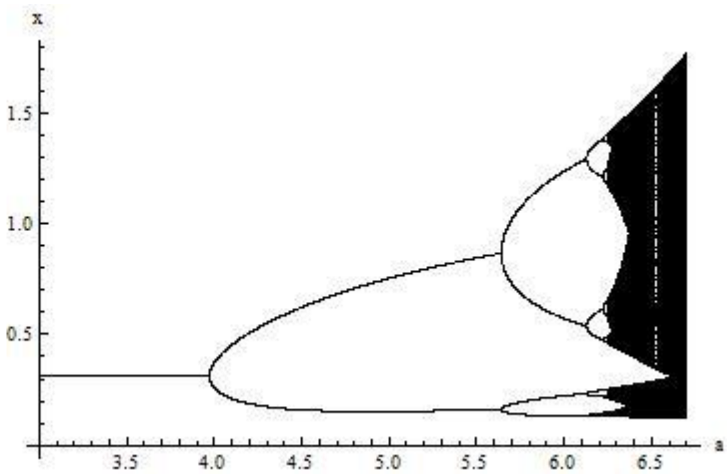

Figure 1: Abscissa represents the control parameter a and the ordinate represents the attractor set corresponding the control parameter.

As discussed in section 2 we have calculated the period doubling bifurcation points of the model (3).The following table gives the value of the bifurcation parameter upto period 1024.

Table 1: Bifurcation parameters at various periods.

SL No	Period	Bifurcation Point
1	1	3.971950227340492638910610208068013920890896306786941680559535849048452788357981747968927142026343790
2	2	5.641947467532494891883905921199693859450629270963613387345747707475761790750845199813300036016814389
3	4	6.128814222101600877120193552121012051413752696635333837522060239223409268378716045163646921383837356
4	8	6.224941048183555277325801814414819645144923160997407253644430373534202379433959828832256427576594618 5
5	16	6.245770284527563837591996749999447524119131635332766985488440030512816267432226273841177661360712882

6	32	6.2502277295678959465303137410040767084340449407506139630861248163178034349836029492364736714656748606
7	64	6.2511826044838604075081058179541697215367305831433470032268532596001630729929171350025126452852679472
8	128	6.251387109555329357062503494433823365133603606733760683995490407331466613073121395944673854332167311
9	256	6.2514309085566484529523964397758981004219351233420396318482182059253473650775582702097657300733951314
10	512	6.2514402889669857026871817562400692468026225646581274925606090069412299618663007177110527354346356902
11	1024	6.2514422979640681911986261879913728870188506313408729719640811948665112033413565624262147219495284318

4. Calculation of accumulation point[6]:

Let $\{A_n\}$ be the sequence of bifurcation points. Using the Feigenbaum constant δ [9,10] we calculate the sequence of accumulation points $\{A_{\infty,n}\}$ by applying the formula $A_{\infty,n}= \frac{A_n - A_{n-1}}{\delta} + A_n$. The values are as follows:

$A_{\infty,1} = 5.999609736168109$
$A_{\infty,2} = 6.233086171642324$
$A_{\infty,3} = 6.24552847064496$
$A_{\infty,4} = 6.2502312691001425$
$A_{\infty,5} = 6.2511823777954945$
$A_{\infty,6} = 6.251387109449789$
$A_{\infty,7} = 6.251430908279093$
$A_{\infty,8} = 6.2514402889614775$
$A_{\infty,9} = 6.251442297963624$
$A_{\infty,10} = 6.251442728229722$

The above sequence converges to the value 6.251442 ….. which is the required accumulation point [2].

4.1. Lyapunov exponent [2]:

For a particular value of "a", if $x_1^*, x_2^*, \ldots, x_n^*$ are n stable periodic points then the Lyapunov exponent is $\lambda = \lim_{n\to\infty} \frac{\log|f'(x_0)| + \log|f'(x_1)| + \ldots\ldots + \log|f'(x_{n-1})|}{n}$, where x_0 is the initial value. We calculate the Lyapunov exponent for the model (3) at very near to the accumulation point.

Table 2: Lyapunov exponent near the calculated Acumulation point:

Value near the calculated accumulation point	Lyapunov exponent
6.25144	-0.000111855
6.251441	-0.000926377
6.2514412	-0.00190568
6.2514418	-0.000657793
6.251443	0.00039843
6.2514431	0.000237588

From the above table we conclude that our experimental period doubling bifurcation parameters are satisfactory.

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