THREE-LOOP EFFECTS IN THE MASSES OF HEAVY FLAVORED MESONS WITHIN A QCD POTENTIAL MODEL

D. K. Choudhury [1,2], Rashidul Hoque* [1,2], B. J. Hazarika [2]
[1] *Department of Physics, Gauhati University, Guwahati-781 014, India*
[2]*Center of Theoretical Studies,Pandu College,Guwahati-781012,India*
*For correspondence. (rashidul999@gmail.com)

Abstract: The static potential between the two heavy quarks is a fundamental quantity in QCD. In lowest order it is described by the Coulomb potential adopted to QCD. While one-loop effects to this potential were known during 1980; analytical information on the corresponding three-loop effects was reported more recently in 2016. In the present paper , we report the results of three-loop effects in the masses of heavy flavored mesons using QCD linear plus Coulomb Cornell potential. For this purpose, we adopt the meson wave functions based on both Dalgarno's Perturbation Theory (DPT) and Variationally Improved Perturbation Theory (VIPT) as have been pursued by us in recent years. We use the numerical expression for higher order effects as reported by Smirnov et al. for our analysis. Detailed comparison is done with Lattice QCD and QCD Sum rules.

Keywords: quantum chromo dynamics; decay constant; meson mass

PACS: 12.39.-x , 12.39.Jh , 12.39.Pn

1. Introduction:

The static potential between the two heavy quarks is a fundamental quantity in QCD [1]. While its one-loop corrections are computed in [2, 3], the corresponding two-loop effects were reported in late 1990's [4, 5, 6]. Three-loop effets have been reported numerically first for fermionic conntribution [7] followed by gluonic counterpart [8, 9]. Analytic three-loop static potential have been discussed only in the year 2016 [9]. These effects are reported in momentum space where the authors set $q^2 = \mu'^2$ as the renormalization scale to suppress the infrared logarithmics. In this work, we make an analysis of three-loop effects in QCD potential model reported by us in [12, 13, 14] in recent years. The model is based on linear cum Coulomb Cornell potential and uses perturbative Quantum mechanical methods like Dalgarno's Perturbation Theory (DPT) [18-22] and Variationally Imoroved Perturbation Theory (VIPT) [23-27]. In this work quantum perturbation approach [18, 19, 23, 26] is used to calculate the approximate analytical forms of heavy flavored mesons. Here either the linear or the Coulomb part of the potential $V(r) = -\frac{4\alpha_s}{3r} + br$ is used as perturbation.

We discuss the theoretical difference between DPT and VIPT as: Out of these two methods (DPT and VIPT), the 2^{nd} one (VIPT) is comparatively new which combines both the Variational method and the stationary perturbation theory. The method was put forwarded by You S. K. et al [23] and later applied to heavy quarks by Atchinson and Dudek [24] for the linear plus Coulomb potential $(\frac{-4\alpha_s}{3r} + br)$. This method substantially reduces the limitations of usual perturbation theory through the use of Variational method over it.

In general a phenomenological potential can be defined as $V(r) = \frac{-4\alpha_s}{3r} + br + c$; where 'c' is to be fitted from experimental values of some observables like masses and decay constants. However, the scale factor 'c' should not appear in the wave function of a meson itself, a feature violeted in application of DPT [19-21]. However the corresponding wave function in VIPT sees such scale factor completely disappears because of its variational techniques[26, 27]. Furthermore, the perturbation expansion effectively becomes more convergent.

The rest of the paper is organised as follows: Section 2 contains formalism, Section 3 contains results and calculations while Section 4 includes the discussion and conclusion.

2. Formalism:

2.1. Three-loop effects:

For the three-loop effect we follow [8] the numerical solution for the potential at three-loop level in momentum space is given by,

$$V(|q|) = -\frac{4\pi C_F \alpha_s(|\overline{q^2}|)}{\overline{q^2}}[1 + (\frac{\alpha_s}{\pi})\left(2.5833 - 0.2778n_f\right) + (\frac{\alpha_s}{\pi})^2\left(28.5468 - 4.1471n_f + 0.0772n_f^2\right) + (\frac{\alpha_s}{\pi})^3\left(209.884 - 51.4048n_f + 2.9061n_f^2 - 0.0214n_f^3\right)] \quad (1)$$

The corresponding expression in co-ordinate space will be,

$$V(r) = -\frac{C_F}{r}\alpha_s(\mu'^2)[1 + (\frac{\alpha_s}{\pi})(2.5833 - 0.2778n_f) + (\frac{\alpha_s}{\pi})^2(28.5468 - 4.1471n_f + 0.0772n_f^2) + (\frac{\alpha_s}{\pi})^3(209.884 - 51.4048n_f + 2.9061n_f^2 - 0.0214n_f^3)] \quad (2)$$

The relationship between the improved strong coupling constant α_V and the standard leading order strong coupling constant (α_s or $\alpha_{\overline{MS}}$) in $\overline{MS}$-scheme at three-loop level is given by:

$$\alpha_V(\frac{1}{r}) = \alpha_s(\mu'^2)[1 + (\frac{\alpha_s}{\pi})(2.5833 - 0.2778n_f) + (\frac{\alpha_s}{\pi})^2(28.5468 - 4.1471n_f + 0.0772n_f^2) + (\frac{\alpha_s}{\pi})^3(209.884 - 51.4048n_f + 2.9061n_f^2 - 0.0214n_f^3)] \quad (3)$$

V-scheme is the standard way of taking into account the higher order effects of QCD,which are expressed as power series in the strong coupling constant $\alpha_{\overline{MS}}$ .The corresponding two-loop static potential in V-scheme is using as three-loop potential defined in [4, 5, 6] as,

$$V(r) = -\frac{C_F \alpha_V(\frac{1}{r})}{r} \quad (4)$$

Here, α_V is the effective strong coupling constant and C_F is the color factor given by, $C_F = \frac{(N_C^2-1)}{2N_C}$, where N_C is the no. of colors. Generally, the quark gluon interaction is characterised by strong coupling constant $\alpha_{\overline{MS}}(q^2)$ in a dimensionally independent $\overline{MS}$- scheme [5, 17, 39].

2.1.1. Wave functions in both Coulombic parent and linear parent options:

We consider the two representative wave-functions of heavy-flavour mesons obtained by Dalgarno's perturbation theory [18, 19, 20, 21, 22] and Variationally Improved Perturbation Theory [23, 24, 25, 26, 27] for Coulomb plus linear potential $V(r) = -\frac{4\alpha_V}{3r} + br$ and use V-scheme to estimate wave-function at origin (WFO). The two normalized wave functions are:

The wave function in DPT with Coulombic parent [19] is given by,

$$\psi_C^D(r) = \frac{N}{\sqrt{\pi a_0^3}}(1 - \frac{1}{2}\mu b a_0 r^2)\exp\frac{-r}{a_0} \quad (5)$$

where, the superscripts D and V represents Dalgarn's method and VIPT respectively and subscripts C and L defines Coulobic parent and linear parent respectively in the wave functions. The normalization constant is

$$N = \sqrt{\frac{1}{1 - 3\mu b a_0^3 + \frac{45}{8}\mu^2 b^2 a_0^6}} \quad (6)$$

and $a_0 = \frac{1}{\mu\alpha}$; $\alpha = \frac{4\alpha_V}{3}$. The value of b is $b = 0.183\ GeV^2$ [19, 22]. μ is the reduced mass of the mesons.

The wave function in DPT with linear parent [18-22] is given by,

$$\psi_L^D(r) = \frac{N}{r}[Ai[\varrho] - B(a_1 + a_2 r + a_3 r^2)] \tag{7}$$

where,$B = \frac{4\alpha_s}{3}$,$\varrho = (2\mu b)^{\frac{1}{3}} r + \varrho_0$, ϱ_0 are the zeroes of Airy Function and,

$$a_1 = \frac{0.8808(b\mu)^{\frac{1}{3}}}{E} - \frac{a_3}{\mu E} + \frac{4W' \times 0.21}{3\alpha_s E} \tag{8}$$

$$a_2 = \frac{ba_1}{c} + \frac{4W' \times 0.8808(b\mu)^{\frac{1}{3}}}{3\alpha_s E} - \frac{0.6535 \times (b\mu)^{\frac{1}{3}}}{E} \tag{9}$$

$$a_3 = \frac{4\mu W' \times 0.1183}{3\alpha_s} \tag{10}$$

with

$$W^0 = E = -(\frac{b^2}{2\mu})^{\frac{1}{3}} \varrho_0 \tag{11}$$

$$W' = 4\pi \int_0^\infty r^2 H' |\psi(r)|^2 dr \tag{12}$$

The VIPT normalized wave function with Coulombic parent [26] corrected upto first order is given by

$$\psi_C^V(\alpha') = N' \psi'(\alpha')[1 - R(1 - \frac{\mu\alpha' r}{2}) \exp^{\frac{\mu\alpha' r}{2}}] \tag{13}$$

Here, α' is a variational parameter.

Note that the ground state wave function $\psi'(\alpha')$ is already normalized and it is given by

$$\psi'(\alpha') = \frac{(\mu\alpha')^{3/2}}{\sqrt{\pi}} \exp^{-\mu\alpha' r} \tag{14}$$

Now following variational method [23-27] we get;

$$E(\alpha') = <\psi' \mid H \mid \psi'> = \frac{1}{2}\mu\alpha'^2 - A\mu\alpha' + \frac{3b}{2\mu\alpha'} \tag{15}$$

where,$A = \frac{4\alpha_V}{3}$,and α_V is the strong coupling constant in V-scheme. Now,minimising ,$\frac{dE}{d\alpha'} = 0$ we get,

$$\alpha'^3 - A\alpha'^2 - \frac{3b}{2\mu^2} = 0 \tag{16}$$

This equation is solved by using $Mathematica7$ and we find the variational parameter α' for different heavy-flavour mesons which is shown in $Table.5$ at three-loop level.

Whereas, normalization constant N' in equation (13) is given as:

$$N' = \sqrt{\frac{K^3}{4\pi(0.25P^2 + 2Q^2)}} \tag{17}$$

with,

$$K = \mu\alpha' \tag{18}$$

$$P = \frac{(\mu\alpha')^{3/2}}{\sqrt{\pi}} \tag{19}$$

$$Q = \frac{4}{3}\sqrt{\frac{\mu}{\alpha'\pi}}(\frac{4\mu\alpha^{'}(\alpha-\alpha^{'})}{27} - \frac{32b}{81\mu\alpha^{'}}) \tag{20}$$

and

$$R = \frac{Q\sqrt{\pi}}{(\mu\alpha')^3/2} \tag{21}$$

The VIPT normalized wave function with linear parent [24, 27] corrected upto first order is given by

$$\psi_L^V = N[\psi_{(10)} + \frac{(2\mu)^{\frac{1}{3}}}{(\varrho_{02}-\varrho_{01})b'^{\frac{2}{3}}}((b-b') < r >_{2,1} -\alpha < \frac{1}{r} >_{2,1})\psi_{20}(r)] \tag{22}$$

where,

$$< r >_{2,1} = \int_0^\infty r A_i[(2\mu b')^{\frac{1}{3}} r - 2.3194] A_i[(2\mu b')^{\frac{1}{3}} r - 4.083] dr \tag{23}$$

and $\alpha = \frac{4\alpha_V}{3}$,also,

$$< \frac{1}{r} >_{2,1} = \int_0^\infty \frac{1}{r} A_i[(2\mu b')^{\frac{1}{3}} r - 2.3194] A_i[(2\mu b')^{\frac{1}{3}} r - 4.083] dr \tag{24}$$

The normalization constant is obtained from,

$$\int_0^\infty 4\pi r^2 |\psi_T|^2 dr = 1 \tag{25}$$

and

$$\psi_{10}(r) = \frac{1}{2\sqrt{\pi r}} A_i[(2\mu b')^{\frac{1}{3}} r + \varrho_{0n}] \tag{26}$$

Here,b' is the variational parameter and ϱ_{0n} are the zeroes of airy function such that $A_i[\varrho_{0n}] = 0$, and is given as [27] :

$$\varrho_{0n} = -[\frac{3\pi(4n-1)}{8}]^{\frac{2}{3}} \tag{27}$$

For different S states few zeroes of the Airy function is listed below.

Zeroes of Airy function for different S-states.

States	ϱ_{0n}
1s(n=1,l=0)	-2.3194
2s(n=2,l=0)	-4.083
3s(n=3,l=0)	-5.5182
4s(n=4,l=0)	-6.782

While dealing with the Airy function as the trial wave-function of variational method with Cornell potential,the main problem is that the wave-function has got a singularity at $r = 0$. The presence of singularity in a wave-function is not new and in QED also such singularities appear. Therefore as discussed in our previous work [18],to calculate the wave-function at the origin,we follow a second method as Quigg [28]. In this method, the wave-function at the origin is found from the condition,$|\psi(0)|^2 = \frac{\mu}{2\pi}\langle\frac{\delta V}{\delta r}\rangle$,and we find the variational parameter b'.

Table 1: Variational parameter for Airy trial function.

Mesons	b'
$D(c\bar{u}/\bar{c}d)$	2.050
$D(c\bar{s})$	1.597
$B(u\bar{b}/d\bar{b})$	1.7269
$B_s(s\bar{b})$	1.2709
$B(\bar{b}c)$	0.558

2.2. Wave function at origin (WFO):

(i) DPT: At origin, r = 0; WFO with Coulombic parent $(Eq^n.5)$ is given by,

$$\psi_C^D(0) = \frac{N}{\sqrt{\pi a_0^3}} \tag{28}$$

Similarly, WFO with Linear parent $(Eq^n.7)$ is given by,

$$\psi_L^D(0) = \sqrt{\frac{\mu b}{2\pi}} \tag{29}$$

(ii) VIPT: At origin, r = 0; WFO with Coulombic parent $(Eq^n.13)$ is given by,

$$\psi_C^V(0) = N'[\frac{(\mu\alpha')^{3/2}}{\sqrt{\pi}} - \frac{4}{3}\sqrt{\frac{\mu}{\alpha'\pi}}(\frac{4\mu\alpha'(\alpha-\alpha')}{27} - \frac{32b}{81\mu\alpha'})] \tag{30}$$

Similarly, WFO with Linear parent $(Eq^n.22)$ is given by,

$$\psi_L^V(0) = \sqrt{\frac{\mu b'}{2\pi}} \tag{31}$$

2.3. The expression of mass (M_P) of pseudo-scalar meson:

The mass of heavy-flavour pseudo scalar meson composing of a light quark q and a heavy quark Q of masses m_q and m_Q is given in [24, 30, 36] as:

$$M_P = m_q + m_Q - \frac{8\pi\alpha_s}{3m_q m_Q}|\psi(0)|^2 \tag{32}$$

where α_s is identified as α_V-the effective strong coupling constant in V-scheme defined by equation (3) and $|\psi(0)|$ is the magnitude of the wave function at origin (WFO) given by equation (28)-(31).

3. Results:

For numerical analysis, We use the usual expression for strong coupling constant in $\overline{MS}$-scheme for lowest order (LO level) is given by [39],

$$\alpha_s(q^2) = \alpha_{\overline{MS}}(q^2) = \frac{4\pi}{\beta_0 ln(\frac{m_Q^2}{\Lambda_{QCD}^2})} \tag{33}$$

Here, m_Q is the mass of the bare heavy quark and Λ_{QCD} is the QCD scale parameter having values 0.292 Gev and 0.210 Gev for $n_f = 4$ and $n_f = 5$ respectively. Taking $m_c = 1.55$ GeV and $m_b = 4.95$ GeV, we found the value of $\alpha_{\overline{MS}}(q^2)$ at c-scale $(n_f = 4)$ and b-scale $(n_f = 5)$ are 0.45 and 0.259 tabulated in Table 2. These values differ from the values of $\alpha_{\overline{MS}}(q^2)$ for $n_f = 4$ and $n_f = 5$ used by us earlier.

Table 2: Values of Λ_{QCD} and $\alpha_{\overline{MS}}(q^2)$ at c-scale and b-scale.

Meson	n$_f$	Λ_{QCD}	$\alpha_{\overline{MS}}(q^2)$
D	4	0.292	0.45
D$_s$	4	0.292	0.45
B	5	0.210	0.259
B$_s$	5	0.210	0.259

3.1. Calculation of effective strong coupling constant $\alpha_V(\frac{1}{r})$ in one-,two- and three-loop effects and their flavour dependence:

Table 3: Values of $\alpha_V(\frac{1}{r})$.

n_f	LO	NLO	NNLO	NNNLO
4	0.45	0.5445	0.67	0.73
5	0.259	0.28	0.297	0.303

Using equation (3), we tabulate the values of $\alpha_V(\frac{1}{r})$ taking into account one-loop (NLO), two-loop (N^2LO) and three-loop (N^3LO) in Table 3 for $n_f = 4$ and $n_f = 5$. It shows that for $n_f = 4$, the enhancement are respectively 21%, 48% and 62% while for $n_f = 5$, the corresponding enhancement are 8% , 14% and 17% respectively. The anti-screening effects of gluons seem to play an important role for $n_f = 5$.

In table 4 and 5, we record the corresponding variation of the parameters α and $\alpha^{'}$ for a few heavy-light mesons.

Table 4: Values of parameter α.

Meson	$\alpha(LO)$	$\alpha(NLO)$	$\alpha(NNLO)$	$\alpha(NNNLO)$
D	0.53	0.72	0.89	0.97
D$_s$	0.53	0.72	0.89	0.97
B	0.345	0.373	0.396	0.404
B$_s$	0.345	0.373	0.396	0.404

Table 5: Values of variational parameter $\alpha^{'}$.

Meson	$\alpha^{'}(LO)$	$\alpha^{'}(NLO)$	$\alpha^{'}(NNLO)$	$\alpha^{'}(NNNLO)$
D	1.73	1.81	1.89	1.93
D$_s$	1.31	1.40	1.49	1.54
B	1.53	1.54	1.55	1.56
B$_s$	1.25	1.26	1.27	1.28

3.2. Calculation of mass:

3.2.1. Coulombic parent:

Following equations (28) , (30) and (32) along with the results of Table 2, Table 3 ,Table 4 and Table 5; we calculated the masses (in GeV) of the four representative heavy-light mesons $D(c\bar{u}/c\bar{d})$, $D_s(c\bar{s})$, $B(u\bar{b}/d\bar{b})$ and $B_s(s\bar{b})$ and tabulated it in Table 6

Table 6: Masses of heavy flavored mesons in GeV.

Meson	Method	M$_P(LO)$	M$_P(NLO)$	M$_P(NNLO$	M$_P(NNNLO)$
D(c$\overline{u}/c\overline{d}$)	VIPT	1.330	1.072	0.810	0.600
D(c$\overline{u}/c\overline{d}$)	DPT	1.8859	1.885	1.883	1.882
D$_s(c\overline{s})$	VIPT	1.746	1.595	1.359	1.217
D$_s(c\overline{s})$	DPT	2.0329	2.032	2.027	2.019
B(u$\bar{b}/d\bar{b}$)	VIPT	5.200	5.187	5.180	5.170
B(u$\bar{b}/d\bar{b}$)	DPT	5.2859	5.2859	5.285	5.285
B$_s(s\bar{b})$	VIPT	5.351	5.344	5.334	5.328
B$_s(s\bar{b})$	DPT	5.4329	5.4329	5.432	5.432

3.2.2. Linear parent:

Using equations (29) , (31) and (32) along with the results of Table 2, Table 3, Table 4 and Table 5; we calculated the masses (in GeV) of the same four respective heavy-light mesons and tabulated it in Table 7.

Table 7: Masses of heavy flavored mesons in GeV.

Meson	Method	M$_P(LO)$	M$_P(NLO)$	M$_P(NNLO$	M$_P(NNNLO)$
D(c$\overline{u}/c\overline{d}$)	VIPT	1.235	1.100	0.920	0.830
D(c$\overline{u}/c\overline{d}$)	DPT	1.827	1.816	1.800	1.792
D$_s(c\overline{s})$	VIPT	1.565	1.467	1.336	1.274
D$_s(c\overline{s})$	DPT	1.979	1.968	1.952	1.946
B(u$\bar{b}/d\bar{b}$)	VIPT	5.174	5.165	5.157	5.155
B(u$\bar{b}/d\bar{b}$)	DPT	5.274	5.273	5.272	5.272
B$_s(s\bar{b})$	VIPT	5.353	5.346	5.241	5.239
B$_s(s\bar{b})$	DPT	5.421	5.420	5.419	5.419

3.2.3. Comparison of our best results with the results of other models and experimental values:

Table 8: Masses of heavy flavoured mesons in GeV.

Meson	M_P(Coul)	M_P(lin)	M_P(Lat[43])	M_P(Q.sum[42])	M_P(Vari[36])	M_P(Exp[35])
D(c$\overline{u}/c\overline{d}$)	1.474 (VIPT) 1.882 (DPT)	1.06 (VIPT) 1.827 (DPT)	1.885	1.87	1.94	1.869$\pm$0.0016
D$_s(c\overline{s})$	1.56 (VIPT) 2.019 (DPT)	1.43 (VIPT) 1.968 (DPT)	1.969	1.97	2.032	1.968$\pm$0.0033
B(u$\bar{b}/d\bar{b}$)	5.00 (VIPT) 5.28 (DPT)	5.87 (VIPT) 5.273 (DPT)	5.283	5.28	5.35	5.279$\pm$0.0017
B$_s(s\bar{b})$	5.334 (VIPT) 5.432 (DPT)	5.346 (VIPT) 5.419 (DPT)	5.366	5.37	5.48	5.366$\pm$0.0024

Following the formalism developed in section 2, we calculated the masses of pseudo-scalar mesons which are shown in Table 6 and Table 7. The input values are, $m_{u/d} = 0.336GeV$, $m_b = 4.95GeV$, $m_c = 1.55GeV$, $m_s = 0.483GeV$ and $b = 0.183GeV^2$. The calculated masses of different heavy-light mesons are now compared with the recent results from lattice QCD [43] and QCD sum rules [42] and the results found in variational mrthod [36] for Gaussian trial wave function and also with recent experimental results [35] in Table 8. It can be easily seen that from the Table 8, our results with DPT for the both option Coulombic parent and linear parent agrees well with those results of other models and available data in V-scheme. Specifically,the mass of B meson estimated in DPT with Coulombic parent is $5.285 GeV$ and in linear parent is $5.273 GeV$ in N^3LO level, which is in excellent agreement with Lattice QCD $(5.283 GeV)$, QCD sum rule $(5.28 GeV)$ and the recent PDG data; $(5.279 \pm 0.0017) GeV$ [35]. Also it is found that the mass of B_s meson estimated in DPT with Coulombic parent is $5.432 GeV$ and in linear parent is $5.419 GeV$ seems to be close agreement with other models and experimental data; 5.366 ± 0.0024. GeV [35]. It is worthwhile to mention that the mass of the B_s meson estimated in VIPT in Coulombic and linear parent are found $5.334 GeV$ at NNLO level and $5.346 GeV$ at NLO level respectively agrees well with experimental PDG data. The pattern is more or less similar in other mesons. For Coulombic parent option, the values of masses from Table 6, in DPT (Dalgarno's perturbation theory) are found $1.882 GeV$ and $2.019 GeV$ for D andD_s mesons at N^3LO level which are in close proximity of experimental values [35] respectively, but in VIPT (Variationally Improved Perturbation Theory) the masses are found $1.217 GeV$ and $1.271 GeV$ respectively , which shows a bad agreement with experimental data available [35]. Therefore, the predicted values of masses M_P are found to be compatible with DPT rather than VIPT. On the the other hand, from the results of Table 7, it is confirmed that when we consider the linear part of the Cornell potential as a parent potential then the DPT and the VIPT show better results for obtaining the masses of the heavy-light pseudo-scalar mesons at N^3LO in both the $\overline{MS}$ and V-scheme. Thus the linear parent seems to be the better option at phenomenological level in the present work.

4. Discussion and conclusion:

1. In this work, it is found that the perturbation theory is suitable at the choice of the scale $m_Q^2 >> \Lambda_{QCD}^2$, where m_Q^2 is the square of the bare quark mass.

2. With fixed α_s, the WFO for VIPT is more than that for DPT and hence M_P is lesser. It is also evident from equation (32) that with the increase in α_s the M_P values are lesser. So, when using V-scheme which is an enhancement scheme for α_s makes VIPT less effective as compared to DPT which indicates that the sensitivity of the running coupling constant is more in DPT over VIPT and this is the reason why DPT shows better resuls in Coulombic parent.

3. From the observation of masses of heavy flavoured mesons , it is found that for the four representative pseudo-scalar mesons $D(c\bar{u}/c\bar{d})$, $D_s(c\bar{s})$, $B(u\bar{b}/d\bar{b})$ and $B_s(\bar{b})$, the former (DPT) [18-22] results are in close agreement with the experimental results over the later (VIPT)[23-27].

4. At a higher mass scale eg. m_Z scale, the strong coupling constant reduces considerably and this is true for any other higher order mass scale like t-quark mass scale. At such higher scale both DPT and VIPT will fail to confirm the data.

5. The results seem to favour V-scheme over $\overline{MS}$-scheme justifying the importance of three-loop effects in our analysis. Further the option of linear parent is found more effective over the Coulombic one.

6. We observed that the value of α_V in V-scheme always exceeds the value of α_s in $\overline{MS}$ -scheme which plays a major role in our analysis.

7. For the linear parent, both VIPT and DPT are found to be consistent.

To conclude, our analysis indicates that the three-loop effects plays an important role in the masses of the pseudo-scalar heavy-flavour mesons.

References:

[1] N.Brambilla et al. Quarkonium Working Group; arXiv:hep-ph/0412158.
[2] W.Fischler,Nicl.Phys.B 129 (1977) 157.
[3] A.Billoire, Phys.Lett.B 92 (1980) 343.
[4] M. Peter, Phys. Rev. Lett.**78**, 602 (1997); arXiv: hep-ph/9610209.

[5] M. Peter, Phys.B 501 (1997); arXiv: hep-ph/9702245.
[6] Y.Schroder, The static potential in QCD to two loops, DESY-98-191, hep-ph/9812205.
[7] Alexander V. Smirnov et al. Fermionic contributions to the three-loop static potential ; PLB 668 (2008) 293; arXiv: hep-ph/0809.1927 v1.
[8] Alexander V. Smirnov et al. Three-loop static potential ; PRL 104 (2010) 112002; arXiv: hep-ph/0911.4742 V2.
[9] Roman N. Lee et al. Phys. Dev. D 94, 054029 (2016).
[10] C.Anzai, Y.Kiyo and Y.Sumino, PRL 104 (2010) 112003.
[11] Michael Melles, Phys. R. D. Vol.62; 074019; Static QCD Potential in co-ordinate spsce with quark masses through two-loops.
[12] D. K. Choudhury and N. S. Bordoloi,Mod.Phys.Lett.A.,Vol. 24, No. 6 (2009) 443-451.
[13] K. K. Pathak and D. K. Choudhury, Chin. Phys. Lett. **28**, 101201 (2011).
[14] D.K.Choudhury and N.S.Bordoloi,Mod.Phys.Lett.A.,Vol. 17, No. 29 (2002) 1909-1913.
[15] Y.Schroder, The static potential in QCD to two loops , DESY-98-191, hep-ph/9812205.
[16] E.Eichten etal.,Phys.Rev.D21(1980)203.
[17] S.J.Brodsky, "Commensurate scale relations and the Abelian correspondence principle",SLAC-PUB-7861(1998).
[18] A.Dalgarno; Stationary Perturbation Theory in Quantum Theory:I Elements: (Ed:D.R.Bates) Academic Press, New York (1961).
[19] D.K.Choudhury and N. S. Bordoloi,International Journal of Modern Physics A, Vol. 15, No. 23 (2000) 3667-3678.
[20] K.K.Pathak,D.K.Choudhury and N. S. Bordoloi, IJMP A (July,2012); Leptonic decay of Heavy-light Mesons in a QCD Potential Model,DOI: 10.1142/S0217751X13500103.
[21] T Das and D.K.Choudhury, Int.J of Mod.Phy A,2016,DOI: 10.1142/S0217751X1650189X.
[22] T Das and D.K.Choudhury and K.K.Pathak, Indian Journal of Phys.(2016) 90:1307-1312,DOI: 10.1007/S12648-016-0866-1.
[23] S K You, K J Jeon, C K Kim and K Nahm, Eur. J. Phys. 19, 179 (1998).
[24] I J R Aitchison and J J Dudek, Eur. J. Phys. 23, 605 (2002).
[25] F M Fernandez, Eur. J. Phys. 24, 289 (2003).
[26] B.J.Hazarika and D.K. Choudhury, Pramana J.Phys.Vol.75, No. 3, September 2010, pp.423-438.
[27] B.J.Hazarika and D.K. Choudhury, Pramana J.Phys.Vol.78, No. 4, pp.555-564, April/2012. hep-ph/1101.1584.
[28] Quigg C and Rosner J L 1979 Phys. Rep.56, 167.
[29] A. Ghataak, S. Lokanathan, Quantum Mechanics, Theory and Applications(TRINITY,New Delhi,2017).
[30] D Griffiths, Introduction to Elementary Particles;John Wiley and Sons, New york(1987),p158.
[31] Halzen and Martin,"Quarks and Leptons",John Wiley and Sons,ISBN:0-471-88741-2,P65.
[32] A.K.Rai,R.H.Parmar and P.C.Vinodkumar,J. Phys. G: Nucl. Part. Phys. 28, (2002).
[33] J.Lahkar, D.K.Choudhury and B.J.Hazarika, arXiv:1902.02079v1.
[34] G. P. Lepage and P. B. Mackenzie, Phys. Rev. D48, 2250 (1993).
[35] C. Patrignani and Particle Data Group, 2016 Chinese Phys. C 40 100001.
[36] J.Lahkar,D.K.Choudhury and B.J.Hazarika,Commun.theo.Phys.,71 (2019) 49-55.
[37] N Isgur and M B Wise, Phys. Lett. B232, 113 (1989).
[38] B.J.Hazarika, K.K.Pathak and D.K.Choudhury., arXiv:1012.4377v3.
[39] Alexandre Deur,Stanley J. Brodsky and Guy F. de Teramond, arXiv: 1604.08082v3.
[40] J.Lahkar,R.Hoque and D.K.Choudhury, Mod.Phys.Lett.A., Vol.34 (2019), 1950106, DOI:10.1142/S0217732319501062.
[41] W.Lucha, et al.,J.Phys.G :Nucl.Part.Phys.38(2011) 105002.
[42] Z.G.Wang, Eur.Phys.J. C 75 (2015) 427, DOI:10.1140/epjc/s10052-015-3653-9.
[43] R.J.Dowdall, et al., HPQCD Collab., arXiv:1207.5749v1.